1 Numerical Methods

1.3 Parabolic Partial Differential Equations (7 units)

Part II Numerical Analysis is useful but not essential, since the required background can readily be found in references [1, 2, 3], and elsewhere.

1 Formulation

For times $0 \leqslant t < \infty$ we wish to solve the diffusion equation

$$\theta_t = \theta_{xx}$$

on the interval $0 \leqslant x \leqslant 1$, with boundary conditions

$$\theta(0,t) = f(t) \quad \text{and} \quad \theta(1,t) = 0 \quad \text{for } 0 \leqslant t < \infty\,, \quad \text{where } f(t) = t(1-t),$$

and with initial condition

$$\theta(x,t) = 0 \quad \text{for} \quad t \leqslant 0\,, \quad 0 \leqslant x \leqslant 1.$$

This is the (non-dimensionalised) initial-value problem for the conduction of heat down a bar when the temperature of one end varies in time. The aim is to study the performance of three simple finite-difference methods applied to this problem, for which the numerical solutions can be compared with an analytic one.

2 Analytic solution

Question 1

(i) To find an analytic solution of the problem first write

$$\theta(x,t) = f(t)(1-x) + \phi(x,t)\,.$$

Next find the governing equation, boundary conditions and initial condition for $\phi(x,t)$. Thence, with justification, solve for ϕ in terms of a Fourier sine series in x.

(ii) Deduce, either from the Fourier sine series or otherwise, that as $t \to \infty$

$$\phi(x,t) \to \alpha(x)t + \beta(x)\,, \tag{1}$$

where the functions $\alpha(x)$ and $\beta(x)$ are to be identified.

(iii) Write a program to compute the analytic solution by summing a finite number of terms of the series, or otherwise.

(iv) Plot θ against x at a few judiciously chosen values of t to illustrate the evolution in time.

(v) How have you satisfied yourself that the solution has been computed to 'sufficient' accuracy?

(vi) Discuss the evolution of the temperature in terms of the physics.

3 Numerical Methods

Divide $0 \leqslant x \leqslant 1$ into N intervals, each of size $\delta x \equiv 1/N$. The aim is to march the solution forward in time for various time steps δt. We consider three schemes.

(i) Approximate θ_t by a forward difference in time and θ_{xx} by a spatial central difference at the current time, which gives the numerical scheme

$$\frac{\theta_n^{m+1} - \theta_n^m}{\delta t} = \left(\delta^2\theta\right)_n^m \equiv \frac{\theta_{n+1}^m - 2\theta_n^m + \theta_{n-1}^m}{(\delta x)^2},$$

where θ_n^m is an approximation to $\theta\,(n\delta x, m\delta t)$.

(ii) Approximate θ_t instead by a central difference in time, so that

$$\frac{\theta_n^{m+1} - \theta_n^{m-1}}{2\delta t} = \left(\delta^2\theta\right)_n^m .$$

In this case you will need scheme (i) in order to make the first step.

(iii) Modify scheme (i) to

$$\frac{\theta_n^{m+1} - \theta_n^m}{\delta t} = \rho\left(\delta^2\theta\right)_n^{m+1} + (1-\rho)\left(\delta^2\theta\right)_n^m$$

with $0 < \rho \leqslant 1$. This is now an *implicit* method, and at each step $(N+1)$ simultaneous equations have to be solved for the θ_n^{m+1}.

Remarks

(a) The matrix of the simultaneous equations is tridiagonal. Therefore the system may be solved quickly and efficiently by exploiting the sparsity. Your code *should make use of the sparsity*, e.g. the matrix should be stored in an efficient way, and needless multiplications by zero avoided. If you are using MATLAB then `help sparse`, `help spdiags` and `help speye` should help.

(b) You can check that aspects of your program are working by setting $\rho = 0$ and comparing with the output of scheme (i).

Question 2 It is convenient to introduce the Courant number $\nu = \delta t/(\delta x)^2$.

(i) First run each finite-difference scheme with $N = 5$ and $\nu = \frac{1}{2}$ and, in the case of scheme (iii), $\rho = \frac{1}{2}$. Plot the solution for representative times. In particular, tabulate and plot the numerical solution θ_n^m, the analytic solution $\theta(n\delta x, m\delta t)$ and the error $\theta_n^m - \theta(n\delta x, m\delta t)$ at $t = 0.1$.

(ii) Next investigate a range of values of your choice for the parameters ν (for all schemes) and ρ (for scheme (iii)) and describe the results. You might like to start by considering $\nu = \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and 1, and $N = 5,\ 20,\ 80$. In the case of scheme (iii) also consider $\delta t = \mu\delta x$ (i.e. $\nu = \mu/\delta x$) for appropriate values of the constants μ and ρ.

(iii) Discuss the accuracy and the stability of each scheme, and how these properties vary with N, ν and ρ. For instance, are your results consistent with the theoretical order of accuracy of each scheme, e.g. see [1, 2, 3]? Statements about accuracy and stability should be supported by *selective* reference to your numerical results, displayed as short tables and/or graphs. Relevant theoretical results should be cited *briefly*.

Comment on, and explain, any interesting features, e.g. do you notice anything about the error in the case of scheme (i) with $\nu = \frac{1}{6}$, scheme (iii) with particular choices of ρ and ν, and scheme (iii) with $\rho = \frac{1}{2}$ and $\delta t = \mu\delta x$?

(iv) Explain, *with justification*, which scheme and parameter values you would recommend to achieve a given level of accuracy using the *least* computing resources. In particular, you should consider the total operation count to achieve a given level of accuracy.

(v) For your recommended scheme and parameter values, demonstrate that the numerical solution tends to the asymptotic limit (1) as $t \to \infty$.

References

[1] Ames, W.F. *Numerical Methods for Partial Differential Equations*, Academic Press.

[2] Iserles, A. *A First Course in the Numerical Analysis of Differential Equations*, CUP.

[3] Smith, G.D. *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, OUP.

