14 General Relativity

14.7 Gravitational Radiation from Point Masses in a Keplerian Orbit (8 units)

This project does not require prior knowledge of either of the Part II courses General Relativity or Cosmology, but does require Part IB Methods and Part IA Dynamics. (You may, however, find it useful to review some of your answers after taking appropriate Part II courses; but the computation may be attempted immediately.)

For nearly Newtonian, slow-motion sources in General Relativity, the following are the formulae for the time-averaged losses due to gravitational radiation for the energy and angular momentum of a gravitating source:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{5c^5}\langle \dddot{I}_{jk}\dddot{I}_{jk}\rangle \tag{1}$$

and

$$\frac{\mathrm{d}J_j}{\mathrm{d}t} = -\frac{2G}{5c^5}\varepsilon_{jkl}\langle \ddot{I}_{ki}\dddot{I}_{il}\rangle. \tag{2}$$

Here i, j and k are the three space indices, the summation convention is used and ε is the alternating tensor. (Note that the coordinate system used is Euclidean, so there is no distinction between upstairs and downstairs indices.) The dot represents a derivative with respect to time t (note the asymmetric time derivatives in (2)). I_{jk} is the trace-free part of the second moment of the mass distribution,

$$I_{jk} = Q_{jk} - \tfrac{1}{3}\delta_{jk}Q_{ii},$$

where Q_{jk} is the second moment of the mass distribution (the moment of inertia tensor):

$$Q_{jk} = \sum_a m^{(a)} x_j^{(a)} x_k^{(a)}$$

where the sum is over the (point) masses. (For continuous mass distributions the sum is replaced by an integral.)

The brackets $\langle \bullet \rangle$ in (1) and (2) denote an average over a suitably large spacetime 4-volume (a more precise definition is given below).

Consider a Keplerian orbit of two point masses m_1 and m_2 in the x–y plane, with semi-major axis a and eccentricity e. Assume the origin is the centre of mass. It is convenient to introduce the total mass $M = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2/M$.

Let d be the distance between the two masses and let ψ be the angle of one of the masses relative to the x-axis (so that the other mass has angle $\psi + \pi$) at a particular point of the orbit; so both d and ψ are functions of time. The first point mass is then at $d_1(\cos\psi, \sin\psi, 0)$ and the second point mass is at $d_2(-\cos\psi, -\sin\psi, 0)$, where $d_1 = d\mu/m_1$ and $d_2 = d\mu/m_2$.

Question 1 Calculate the Q_{jk} and hence the I_{jk} as functions of d, μ and ψ. Simplify your answer and write it in matrix form.

For Keplerian motion the orbit equation is

$$d = \frac{a(1-e^2)}{1 + e\cos\psi}$$

and the angular velocity is given by

$$\dot{\psi} = \frac{\sqrt{GMa(1-e^2)}}{d^2}.$$

In equations (1) and (2), assume $\langle \bullet \rangle$ denotes an average in time over one orbit. For a function w of ψ this means that

$$\langle w \rangle = \frac{1}{T}\int_0^T w\big(\psi(t)\big)\,\mathrm{d}t = \frac{1}{T}\int_0^{2\pi} \frac{w(\psi)}{\dot{\psi}}\mathrm{d}\psi$$

where

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

is the period of the orbit.

Assume that a and e do not change appreciably in one orbit, so that their time derivatives can be ignored in the averages.

Question 2 Note that $J_1 = J_2 = 0$ (i.e., the x and y components of $\mathbf{J}$ vanish). Show that

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = -\frac{32G^4\mu^2M^3}{5c^5a^5}f(e) \tag{3}$$

where

$$f(e) = \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}$$

and that

$$\left\langle \frac{\mathrm{d}J_3}{\mathrm{d}t} \right\rangle = -\frac{32G^{7/2}\mu^2M^{5/2}}{5c^5a^{7/2}}g(e) \tag{4}$$

where

$$g(e) = \frac{1+\frac{7}{8}e^2}{(1-e^2)^2}.$$

For Keplerian orbits we have

$$E = -G\frac{\mu M}{2a}$$

and

$$J_3 = \sqrt{G\mu^2Ma(1-e^2)}$$

(J_3 is usually denoted L).

Question 3 For Keplerian orbits prove that

$$\left\langle \frac{\mathrm{d}a}{\mathrm{d}t} \right\rangle = -\frac{64G^3\mu M^2}{5c^5a^3}f(e),$$

and that

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = -\frac{304G^3\mu M^2}{15c^5a^4}h(e)$$

where

$$h(e) = \frac{(1+\frac{121}{304}e^2)e}{(1-e^2)^{5/2}}.$$

Under what conditions is it true that the time derivatives of a and e (as just calculated) can be ignored in the averages in (1) and (2) (as was assumed above in deriving (3) and (4))? (This is a consistency check.)

Question 4 Find a as a function of time in the case when $e \equiv 0$ (i.e., a circular orbit).

Question 5 The WUMa eclipsing binary star system has $m_1 = 0.77M_\odot$ and $m_2 = 0.56M_\odot$ (the Sun has mass $M_\odot \approx 2\times10^{30}\,\mathrm{kg}$), with period $T = 0.33\,\mathrm{days}$. This determines the initial value of a but not of e, so we consider the situation as a function of e.

Using a computer, calculate how much power is being radiated (now) gravitationally for $e = 0$, 0.5 and 0.95. Compare the results with the electromagnetic output of the Sun (about $4 \times 10^{26}\,\mathrm{J\,s^{-1}}$).

Write a program to determine how long it will take, as a function of initial eccentricity e_0, before a decays to zero for WUMa. Use this program to calculate the decay time for initial values $e_0 = 0.00, 0.05, 0.10, \ldots, 0.95$. Compare the results with the present age of the universe (13.7×10^9 years).

Question 6 Write a program to calculate the initial semi-major axis a_0 that a binary must have in order for the inspiral time to be equal to the age of the universe. Use this to calculate a_0 (in units of the solar radius $R_\odot \approx 7 \times 10^8\,\mathrm{m}$) for a binary consisting of a solar mass star and giant gas planet of mass $0.001M_\odot$ (about a Jupiter mass) with $e_0 = 0.00, 0.05, 0.10, \ldots, 0.95$.

Hot Jupiters are a class of exoplanets with masses similar to Jupiter, but orbiting at $< 0.5\,\mathrm{AU}$ (where $1\mathrm{AU} \approx 1.5 \times 10^{11}\,\mathrm{m}$). Since Jupiter-like planets are expected to form at around $5\,\mathrm{AU}$, using your above results comment on whether gravitational radiation is a possible mechanism for the Hot Jupiters to have subsequently migrated to their current location.

