14 General Relativity

14.6 Isolating Integrals for Geodesic Motion (8 units)

This project assumes material taught in the Part II course General Relativity. Some of the calculations may be done more simply using a Computer Algebra System (CAS) such as Mathematica or Maple, or the symbolic toolbox in MATLAB. Throughout we use geometrical units with $c = G = 1$.

1 Geodesic Motion in Axisymmetric Spacetimes

A general axisymmetric metric can be written in the form

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{\phi\phi}\mathrm{d}\phi^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 \tag{1}$$

where the metric components are functions of the coordinates r and θ only.

Question 1 By considering the Euler-Lagrange equations for t and ϕ of the geodesic action

$$\mathcal{S} = \int \sqrt{g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}}\,\mathrm{d}\tau \tag{2}$$

where the affine parameter τ is the proper time along the geodesic, or otherwise, show that

$$\begin{aligned} E &= g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau} + g_{t\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \\ L_z &= -\left(g_{t\phi}\frac{\mathrm{d}t}{\mathrm{d}\tau} + g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) \end{aligned} \tag{3}$$

are constants of geodesic motion in any axisymmetric spacetime (1). Hence, derive the mass conservation integral

$$g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + g_{\theta\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 = -V_{\mathrm{eff}}(r,\theta,E,L_z) \tag{4}$$

where the *effective potential*, V_{eff}, should be found in terms of E, L_z and the metric components.

The effective potential defines the allowed regions of geodesic motion for a particular choice of the energy, E, and angular momentum, L_z. Motion is only possible where $V_{\mathrm{eff}} \geqslant 0$.

The Kerr metric has components

$$\begin{aligned} &g_{tt} = 1 - \frac{2\,m\,r}{\Sigma}, \qquad g_{t\phi} = \frac{2\,a\,m\,r\,\sin^2\theta}{\Sigma}, \qquad g_{\phi\phi} = -\left(\Delta + \frac{2\,m\,r\,(r^2+a^2)}{\Sigma}\right)\sin^2\theta, \\ &g_{rr} = -\frac{\Sigma}{\Delta}, \qquad g_{\theta\theta} = -\Sigma \end{aligned} \tag{5}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2\,m\,r + a^2$, m is a constant (the mass of the black hole) and a is another constant (the spin parameter of the black hole).

Question 2 Using a CAS, or otherwise, derive expressions for the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{im}\left(g_{jm,k} + g_{km,j} - g_{jk,m}\right) \tag{6}$$

for the Kerr metric (5). You can present these results in your write-up in the form of a printout from a CAS worksheet.

Programming Task: Write a program to numerically integrate the second order timelike geodesic equations

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} \tag{7}$$

for the Kerr metric (5). Do not make use of the first integrals derived above (3)–(4), but write the four second order equations as a set of eight coupled first order equations. We will use the first integrals to verify the numerical accuracy of the integrations.

2 Geodesic motion in the Schwarzschild metric

The Schwarzschild metric may be obtained by setting $a = 0$ in the Kerr metric (5).

Question 3 What are the non-zero Christoffel symbols for the Schwarzschild metric? Take $m = 1$ and find the zeros of the effective potential (4) in the equatorial plane, $\theta = \pi/2$, for the case $E = 0.97$, $L_z = 4$. Hence determine the allowed range of radii, r, of bound orbits in the equatorial plane. Then, using your geodesic code, do the following

a Take initial conditions $r = 15$, $\theta = \pi/2$, $\mathrm{d}r/\mathrm{d}\tau = 0$ and the value of $\mathrm{d}\theta/\mathrm{d}\tau$ determined from the effective potential (4). Plot the coordinates, $(t,\, r,\, \theta,\, \phi)$, of the particle as a function of τ over several orbits. Check that the three conservation laws (3)–(4) are satisfied at a reasonable level of numerical accuracy.

b For the same choice of E and L_z, take a range of initial conditions that lead to bound motion (e.g., consider initial conditions in the equatorial plane with $\mathrm{d}r/\mathrm{d}\tau = 0$ and a range of values of $r(0)$). Output the values of r and $\mathrm{d}r/\mathrm{d}\tau$ every time the orbit crosses the equatorial plane, $\theta = \pi/2$, with $\mathrm{d}\theta/\mathrm{d}\tau > 0$. Plot these values on a graph, with r on the horizontal axis, and $\mathrm{d}r/\mathrm{d}\tau$ on the vertical axis. What do you notice?

c Experiment with a few different values of E, L_z and initial conditions.

You have plotted a Poincaré map for these orbits. If the Poincaré map of an orbit is a closed curve it indicates the possible existence of an extra isolating integral for the motion.

3 Geodesic motion in the Kerr metric

We now consider $a \neq 0$ in the Kerr metric (5).

Question 4 Take $a = 0.9$, $E = 0.95$ and $L_z = 3$, and use the effective potential to find the allowed range of r_0 for which the initial conditions $\theta = \pi/2$, $r = r_0$ and $\mathrm{d}r/\mathrm{d}\tau = 0$ lead to bound motion. Plot a Poincaré map as described above for a range of initial conditions of this type. Is the result similar to what you saw for the Schwarzschild metric?

Question 5 Show that the quantity

$$Q = (aE \sin\theta - L_z \mathrm{cosec}\theta)^2 + (r^2 + a^2 \cos^2\theta)^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \delta\, a^2 \cos^2\theta \tag{8}$$

is conserved for geodesic motion in the Kerr metric, where δ is a numerical constant that should be determined. You may use a CAS to help demonstrate this, but should include evidence of the calculation. What does Q become in the limit $a = 0$, i.e., for the Schwarzschild metric? Provide a physical interpretation if possible.

References

[1] Chandrasekhar, S.; *The Mathematical Theory of Black Holes*; Clarendon Press: Oxford; 1992.

[2] D'Inverno, R.; *Introducing Einstein's Relativity*; Clarendon Press: Oxford; 1992.

[3] Goldstein, H., Poole, C. & Safko, J.; *Classical Mechanics*, third edition, Pearson Education International: New Jersey; 2002.

