12 Nonlinear Dynamics & Dynamical Systems

12.6 Chaos and Shadowing (10 units)

Familiarity with the Part II Dynamical Systems course would be very helpful for this project, which is concerned with the behaviour of nonlinear maps and uses concepts and tools from nonlinear dynamics.

1 Introduction: dynamical systems, chaos and shadowing

This project considers issues that arise in the numerical solution of dynamical systems which display complicated 'chaotic' behaviour. We first consider the discrete-time case, defining Lyapunov exponents which measure the rate at which nearby points separate under iteration. Then we discuss how 'noisy' trajectories of an iterated map, where the 'noise' arises through numerical errors, are actually close to true trajectories of the system - this property is known as 'shadowing'. Finally the project considers a continous-time (ODE) example of complicated motion motivated by celestial mechanics.

Let D be a closed bounded subset of $\mathbb{R}^m$, and let $F(\mathbf{x})$ be a continuously differentiable map from D to itself. A major task in dynamical systems is to characterise the behaviour of points under repeated iteration of the map F. We call the sequence of points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ constructed by setting $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$ the *trajectory* from the initial condition $\mathbf{x}_0$. The standard notation for the repeated composition of F is to let F^n denote the n-fold composition of F with itself, i.e. $\mathbf{x}_n = F(\mathbf{x}_{n-1}) = F^2(\mathbf{x}_{n-2}) = \cdots = F^n(\mathbf{x}_0)$.

In many situations the rate at which nearby trajectories separate from each other is of interest. This can be characterised by the Lyapunov exponents $\lambda(\mathbf{x}_0, \mathbf{v})$, defined to be the asymptotic rate of divergence of trajectories with initial conditions $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{v}$, where $\mathbf{v}$ is a small perturbation from $\mathbf{x}_0$:

$$\lambda(\mathbf{x}_0, \mathbf{v}) = \lim_{n\to\infty} \lim_{\epsilon\to 0} \frac{1}{n} \log \frac{\|F^n(\mathbf{x}_0 + \epsilon\mathbf{v}) - F^n(\mathbf{x}_0)\|}{\|\epsilon\mathbf{v}\|} \tag{1}$$

Under the conditions given above it can be shown that the limit exists. For a given $\mathbf{x}_0$ there will in general be m (possibly non-distinct) values of $\lambda(\mathbf{x}_0, \mathbf{v})$ as we choose different vectors $\mathbf{v}$ – divergence occurs at different rates in directions corresponding to the different eigenvectors of the Jacobian matrix DF evaluated at $\mathbf{x}_0$. The formula above for $\lambda(\mathbf{x}_0, \mathbf{v})$ will give the largest positive Lyapunov exponent of the system for almost all choices of the vector $\mathbf{v}$. We denote the largest positive Lyapunov exponent by $\lambda(\mathbf{x}_0)$, or simply by λ. If $\mathbf{x}_0$ is a fixed point then the Lyapunov exponents are simply the (real parts of the) Floquet multipliers, so in some sense the idea of a Lyapunov exponent developed above is a generalisation of the idea of a Floquet multiplier to arbitrary trajectories.

For the purposes of this project we will define a map to be chaotic if it appears that $\lambda(\mathbf{x}_0) > 0$ for almost all $\mathbf{x}_0$, so that in general neighbouring points will separate exponentially.

1.1 A map on a square

Here we consider a 2-dimensional (area preserving) map on the unit square $(x, y) \in [0, 1]^2$. Given some initial condition (x_0, y_0), we define

$$x_{n+1} = x_n + \frac{K}{2\pi}\sin(2\pi y_n) \quad (\text{mod } 1) \tag{2}$$

$$y_{n+1} = y_n + x_{n+1} \quad (\text{mod } 1) \tag{3}$$

Note that here(mod 1) means that the map is restricted to the unit square.

Question 1 For $K = 3$, generate a set of double precision pairs (x_n, y_n), $1 \leqslant n \leqslant 1000$, for a range of choices of (x_0, y_0). Some suggested choices of initial conditions are : (0.5,0.5); (10^{-8}, 0); (0.1,0.5); (0.8,0.6); (0.3521,0.424).

Plot the distribution of your points (x_n, y_n). Describe the structure of the phase portrait. What are the fixed points of the map? Describe how the different features of the map change with K.

1.2 Local Chaos

In contrast to the asymptotic quantity $\lambda(\mathbf{x}_0)$ as defined above, a possibly more useful quantity is the local Lyapunov exponent, $\lambda_l(\mathbf{x}_0)$, defined as

$$\lambda_l(\mathbf{x}_0) = \lim_{\mathbf{\Delta}\to\mathbf{0}} \frac{1}{N}\sum_{n=0}^{N} \log \frac{\|\mathbf{x}_{n+1} - \mathbf{x}_{n+1}^{(\mathbf{\Delta})}\|}{\|\mathbf{x}_n - \mathbf{x}_n^{(\mathbf{\Delta})}\|} \tag{4}$$

where $\mathbf{x}_n$ is the n^{th} iterate of $\mathbf{x}_0$, $\mathbf{x}_n^{(\mathbf{\Delta})}$ is the n^{th} iterate of $\mathbf{x}_0 + \mathbf{\Delta}$, and N is a suitable finite number of iterations of the map. For infinitesimal perturbations $\mathbf{\Delta}$, $\lambda_l(\mathbf{x}_0) > 0$ indicates a local expansion of trajectories starting near $\mathbf{x}_0$.

Note that N should be chosen neither too small, nor too large. In practice, you might also want to discard the first few terms of this sum in your numerical calculations.

Question 2 For $K = 3$, find the maximum local Lyapunov exponent for different initial conditions (x_0, y_0) and small perturbation $\|\mathbf{\Delta}\| \ll 1$, using the Euclidean norm in equation (4). What value of N did you use? How did you decide? What is your estimate of the global maximum Lyapunov exponent?

Question 3 We can define a different "Lyapunov exponent", $\lambda_2 = \log_2 e^{\lambda_l}$. Why, when doing binary arithmetic, might λ_2 be more interesting than λ? What is your interpretation of the information λ_2 provides? Given that the calculations here are done to 16 significant figures (or to whatever precision achieved by the code you have used), what would you expect the number of iterations required to be before the results obtained become meaningless? How does your answer compare with what you found numerically?

1.3 Shadowing

Numerical calculations introduce round-off and truncation errors into iteration. For chaotic maps, such as the 2D map above this introduces an effective error at each iteration; this is in some sense equivalent to the explicit perturbation in the initial conditions we considered above.

Since a large class of interesting problems is reducible to iterating nonlinear, and chaotic, maps, it is of some interest to consider whether any numerical calculation can be said to follow the "true" trajectory of such systems.

Here we will consider the simple example used above, assuming the "true" trajectory is given by a double precision calculation of the trajectory, while a single precision calculation provides a "noisy" trajectory.

For some nonlinear systems it is possible to define a "shadow" trajectory to a noisy trajectory (obtained by adding a small perturbation), such that the shadow trajectory is a "true" trajectory of the system, and the "shadow distance" (initially the perturbation) of the shadow trajectory from the noisy trajectory is bounded ([1], [2], [5]). In 2D when there exists one *unstable* (expanding) direction and one *stable* (contracting) direction, it has been proved that, for sufficiently small perturbations, "shadow" trajectories can exist for arbitrarily long times. In many other systems it is still possible to define a "shadow" trajectory for a finite time.

> **Question 4** Let L_n be the Jacobian matrix of the map at iteration n. Construct and write down explicitly the four elements of the Jacobian matrix of the standard map above.

Define $e_{n+1} = p_{n+1} - f(p_n)$, where e_n is the error iterating the map, f, on the vector p by one step. We want to construct a correction term, Φ_n, such that

$$\tilde{p}_n = p_n + \Phi_n \tag{5}$$

defines a "shadow" orbit of p, i.e. $\{\tilde{p}_n\}$ is a true orbit of the dynamical system.

Solving for Φ, we find:

$$\Phi_{n+1} = f(\tilde{p}_n) - e_{n+1} - f(p_n). \tag{6}$$

For Φ_n small, we can expand $f(\tilde{p}_n)$ to linear order, and

$$\Phi_{n+1} = L_n \Phi_n - e_{n+1}. \tag{7}$$

At each iteration, small perturbations along the contracting direction will decay exponentially forward in time, while small perturbations in the expanding direction will grow exponentially forward in time. The reverse will happen when evolving backwards in time.

We therefore want to find basis vectors u_n, s_n aligned with the directions defining the maximum expansion and contraction of the local volume of phase space at step n. We can construct u_n, s_n by iterating the equations:

$$u_{n+1} = \frac{L_n u_n}{||L_n u_n||} \tag{8}$$

and

$$s_{n+1} = \frac{L_n s_n}{||L_n s_n||}. \tag{9}$$

That is, take some initial u_0, s_0 (eg. $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$), and a vector $p_0 = (x_0, y_0)$. Iterate u_n forwards, i.e. start with your u_0 and iterate equation (8) forward until it has converged to the local direction of expansion. To construct s_n, do the same, but take the initial s_N for some finite (not too big nor too small) number of iterations of the map, and iterate s_n backwards to find s_0. You will want $N \gg 1$ and $N \ll N_c$, where N_c was the number of iterations at which the sum in equation (4) needed to be truncated.

This procedure will naturally converge onto the direction of maximum expansion when going forward in time, because the term corresponding to the maximum eigenvalue will become dominant for sufficiently large n. Conversely, when going backwards in time the term corresponding to the smaller eigenvalue will become dominant, because it will be proportional to the inverse of a small quantity.

Clearly, since u_n, s_n span the phase space, we can write

$$\Phi_n = \alpha_n u_n + \beta_n s_n \tag{10}$$

and

$$e_n = \eta_n u_n + \xi_n s_n \tag{11}$$

for some α, β, η, ξ.

Using equation (7) we find

$$\alpha_{n+1} u_{n+1} + \beta_{n+1} s_{n+1} = L_n(\alpha_n u_n + \beta_n s_n) - (\eta_{n+1} u_{n+1} + \xi_{n+1} s_{n+1}). \tag{12}$$

> **Question 5** Substitute equations (8) and (9) into equation (12) to find a recursion relation for α_n, β_n. As before, solve for the α_n by forward iteration from $n = 0$ and for the β_n by backward iteration from $n = N$ for some suitable, fixed N.

You now have a constructed shadow map of the trajectory.

> **Question 6** Integrating the standard map in double precision, from some known initial condition, with a known error, show that the shadow map of the erroneous initial conditions follows the true trajectory within some shadow distance. If necessary, iterate the shadowing to get a more closely shadowed orbit.
>
> Now integrate your initial condition with single precision (introducing some, in principle unknown) error per iteration, and construct the corresponding double precision shadow trajectory.
>
> Plot your trajectories and comment. (Finding and plotting such trajectories can be tricky!)

Note that shadowing does not always work. A trivial counter-example is provided by the one dimensional logistic map $f(x) = 1 - 2x^2$, $x \in (-1, 1)$.

Near $x = 0$, no true orbit can shadow general noisy orbits, as noise in $f(x)$ may take the map out of the domain and iterating the subsequent trajectory will take x to $-\infty$.

1.4 Application: the Sitnikov Problem

It is known that the N-body problem, of $N > 2$ bodies moving under their own mutual gravitational attraction only, is chaotic.

Here we consider a well known special case of the restricted three-body problem, where one of the bodies has zero mass. In this particular problem, known as the Sitnikov problem ([4], [3]), the motion of the zero mass is restricted to the z axis, defined by the normal to the plane of motion of the two massive bodies, through the center of mass. The two massive bodies move on Keplerian ellipses with eccentricity $\epsilon \in [0, 1]$ around their centre of mass.

Without loss of generality, we consider the two massive bodies to have masses, $M_1 = M_2 = 1/2$. We are interested in bound motion, with semi-major axis $a = 1$.

The motion of the two massive bodies is uniquely described by their elliptic orbit (the phase is irrelevant to the dynamics we are interested in, by rotational symmetry). The separation of the massive bodies from the center of mass is $r(t) = (1 - \epsilon \cos t) + O(\epsilon^2)$, .

We want to consider the motion of the third, zero mass body on the z-axis. Define $v = dz/dt$, then

$$\frac{dv}{dt} = -\frac{z}{(z^2+1)^{3/2}} - \frac{3z\epsilon\cos(t+t_0)}{(z^2+1)^{5/2}}. \tag{13}$$

The equations of motion may be integrated numerically using a high order integrator, such as the Runge–Kutta scheme, given some initial conditions. Without loss of generality, we choose initial conditions $t_0 = 0, z(0) = 0,\ v(0) = v_0$.

> **Question 7** Write down the energy of the third mass, i.e. $\lim_{m\to 0}(E/m)$ and solve for the critical velocity, v_c, for which the energy is zero. Write down $z(t)$ for $\epsilon = 0$. Write down the Jacobian matrix of this map.
>
> It is useful to define the initial velocity as some multiple of v_c. We are interested in (initially) bound motion, so $v_0 \leqslant v_c$.
>
> For $\epsilon = 0.03, 0.04, 0.05$ and $v_0/v_c = 0.92, 0.94, 0.96$, plot $z(t)$ vs t. Comment.

In continuous time we can define a (maximum) Lyapunov exponent exactly analogous to the discrete-time case:

$$\lambda(\mathbf{z}_0) = \lim_{t\to\infty}\lim_{\epsilon\to 0}\frac{1}{t}\log\frac{\|\phi_t(\mathbf{z}_0+\epsilon\mathbf{w}) - \phi_t(\mathbf{z}_0)\|}{\|\epsilon\mathbf{w}\|} \tag{14}$$

for almost all choices of perturbation $\mathbf{w}$, where $\mathbf{z} = (z, \dot{z})$ and ϕ_t denotes the evolution operator defined by integrating the ODEs forwards in time.

> **Question 8** As before, construct a trajectory in $(z, \dot{z})$ space with an initial "error", δ, and integrate the true and erroneous trajectories for a chosen value of $v_0 \approx 0.95v_c$.
>
> Estimate numerically the Lyapunov exponent of the mapping for the different cases. Is the motion chaotic?

> **Question 9** Using the method discussed in the previous section, construct a shadow trajectory for the zero mass body, and compare "true" trajectories integrated with double precision arithmetic, with the corresponding "shadow" trajectories integrated from the same initial condition with single precision arithmetic.
>
> Comment on the integrability of the N-body problem. Do you think numerical integrations of N-body systems are reliable – or can be made reliable – in some sense?

References

[1] Bowen, R, 1975 *J. Diff. Eqns.*, 18, 333.

[2] Grebogi, C., Hammel, S.M., Yorke, J.A., Sauer, T., 1990 *Phys. Rev. Lett.*, 65, 1527

[3] Liu, J., Sun, Y.-S., 1990 *Cel. Mech. and Dyn. Astro.*, 49, 285.

[4] Marchal, C., 1990 *The Three-Body Problem*, Elsevier (Oxford).

[5] Quinlan, G.D., Tremaine, S., 1992 *MNRAS*, 259, 505.

