2.3 Non-Euclidean Geometry

Some of the material from this project is covered in the IB Geometry course (and in the given reference).

1 Introduction

The aim of this project is to create some pictures that illustrate the action of certain groups of isometries on the sphere, and on the disc model of the hyperbolic plane.

2 Spherical geometry

We work in the extended complex plane $\mathbb{C} \cup \{\infty\}$. In this project a *spherical line* will be either a straight line through the origin, or a circle of the form

$$\{z \in \mathbb{C} : |z - a|^2 = |a|^2 + 1\}$$
(1)

where $a \in \mathbb{C}$. In fact, under stereographic projection, a spherical line corresponds to a great circle on the unit sphere. Moreover stereographic projection preserves angles. A spherical triangle is obtained by joining points $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$ by spherical lines.

Question 1 If the spherical line (1) passes through distinct points z_1 and z_2 , find a formula for a in terms of z_1 and z_2 . Write a program that given $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$, draws and fills in the spherical triangle with vertices z_1, z_2, z_3 .

How do spherical lines meet the unit circle |z| = 1?

The second cosine rule states that if a spherical triangle has side lengths a, b, c and internal angles α, β, γ (the side of length a being opposite the vertex with angle α) then

$$\cos\alpha + \cos\beta\cos\gamma = \sin\beta\sin\gamma\cos a. \tag{2}$$

The side lengths here are "spherical distances" where for example the spherical distance between 0 and z is $2 \tan^{-1} |z|$.

3 Hyperbolic geometry

We work in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. A hyperbolic line is (the intersection of D and) either a straight line through the origin, or a circle of the form

$$\{z \in \mathbb{C} : |z - a|^2 = |a|^2 - 1\}$$
(3)

where $a \in \mathbb{C}$ with |a| > 1. Note that (1) and (3) differ by a change of sign. A hyperbolic triangle is obtained by joining points $z_1, z_2, z_3 \in D$ by hyperbolic lines.

Question 2 Write a program that given $z_1, z_2, z_3 \in D$, draws and fills in the hyperbolic triangle with vertices z_1, z_2, z_3 .

How do hyperbolic lines meet the unit circle |z| = 1?

The second cosine rule states that if a hyperbolic triangle has side lengths a, b, c and internal angles α, β, γ (the side of length a being opposite the vertex with angle α) then

$$\cos\alpha + \cos\beta\cos\gamma = \sin\beta\sin\gamma\cosh a. \tag{4}$$

The side lengths here are "hyperbolic distances" where for example the hyperbolic distance between 0 and z is $2 \tanh^{-1} |z|$.

4 Regular polygons

A spherical or hyperbolic n-gon is obtained by joining n vertices by spherical or hyperbolic lines.

Question 3 For which integers n does there exist r > 0 such that the spherical ngon with vertices at the roots of $z^n = r^n$ has internal angle $2\pi/3$? What happens if we replace 'spherical' by 'hyperbolic'? [Hint: Work out a formula for the area of an n-gon, by subdividing into triangles and quoting a result from the Geometry course.]

Use your programs from Questions 1 and 2 to illustrate how these *n*-gons may be subdivided into 2n triangles each with internal angles $\pi/2$, $\pi/3$ and π/n . [Hint: Use the formulae (2) and (4) to find the vertices.]

5 Symmetry groups and tesselations

Let p be an odd prime. Let SL(2, p) be the finite group of all 2×2 matrices whose entries are integers mod p, and have determinant 1. Let PSL(2, p) be the quotient of SL(2, p) by the subgroup $\{\pm I\}$ of order 2. It can be shown that PSL(2, p) is generated by

$$\sigma_1 = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_2 = \pm \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \sigma_3 = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
(5)

Inversion in a Euclidean circle C is the unique map of the form

$$z\mapsto \frac{a\overline{z}+b}{c\overline{z}+d}$$

that fixes each point of C. For example inversion in the unit circle is $z \mapsto 1/\overline{z}$.

Let L be a spherical or hyperbolic line. We define *reflection in* L to be either the Euclidean reflection (if L is a straight line) or inversion in L (if L is a Euclidean circle).

Let Δ_0 be one of the triangles in Question 3, where now n = p is a prime. Let R_1, R_2, R_3 be the reflections in the sides of Δ_0 opposite the vertices with internal angles $\pi/2, \pi/3$ and π/p (in that order).

Question 4 Describe the transformations $S_1 = R_2R_3$, $S_2 = R_3R_1$ and $S_3 = R_1R_2$ geometrically. Show that they have the same orders as the group elements in (5).

Let $g \in PSL(2, p)$ and let Δ be a triangle. The pair (g, Δ) is *admissible* if it is obtained from $(\pm I, \Delta_0)$ by a sequence of operations replacing (g, Δ) by $(\sigma_i g, S_i \Delta)$ for some i = 1, 2 or 3. Note that we can list the elements of PSL(2, p) by starting with $\pm I$ and then repeatedly multiplying by the generators σ_1, σ_2 and σ_3 until no new elements are found.

Question 5 Let p = 3. For each $g \in PSL(2, p)$ find a triangle Δ such that (g, Δ) is admissible. Use your programs to plot these triangles Δ . How many triangles are there in total? Do the triangles ever overlap? Could we obtain any more triangles by applying the transformations S_1 , S_2 and S_3 ? Repeat the question for p = 5 and p = 7.

Question 6 How do your pictures for p = 3 and p = 5 relate to the Platonic solids? To which other well known groups are PSL(2,3) and PSL(2,5) isomorphic, and how is this suggested by your pictures?

Reference

P.M.H. Wilson, Curved spaces, CUP 2008.