2.3 Curves in the Complex Plane

This project uses material found in both the Complex Methods and Complex Analysis courses.

1 Introduction

A *curve* γ is a continuous map from the closed bounded interval $[0, 2\pi]$ (or $[0, 1]$ or even $[a, b]$) to $\mathbb{C}$. We say that γ is *smooth* if $\gamma(t) = (u(t), v(t))$ for u, v functions that are infinitely differentiable and γ is *closed* if $\gamma(0) = \gamma(2\pi)$.

Suppose that $w = f(z)$ is a complex polynomial in z and we take $r > 0$. If C_r is the circle of radius r then it is the case that $f(C_r)$ is a smooth closed curve in the w-plane. Curves that are generated in this way have a number of interesting properties that will be investigated in this project.

2 Complex roots of $f(z)$

In this section you will find the complex roots of the polynomial

$$f_1(z) = z^3 + z^2 + (5 - 4i)z + 1 - 8i \tag{1}$$

and of its first derivative $f_1'(z)$.

Programming Task: Write a program that, given a polynomial $f(z)$, plots a graph of $f(C_r)$ and computes the coordinates and modulus of the closest point on $f(C_r)$ to $0 + 0i$. Your program should prompt you to enter a value for r.

Question 1 Using your program find the three roots of $f_1(z)$ to three significant figures and record the roots in your write-up, together with the corresponding values of r. There is no need for your program to automate the search for an r such that $\min[|f(C_r)|] = 0$ — trial and improvement is an adequate method. Nevertheless, you may find it helpful to include an option to carry out the search automatically.

Taking your output for the roots, how can you justify that these answers are indeed correct to three significant figures?

Question 2 Write down the first derivative $f_1'(z)$ of $f_1(z)$ and use your program to find the two roots of $f_1'(z)$ to three significant figures. Record the roots in your write-up, together with the corresponding values of r. Again, how is your answer justified?

3 Images $f(C_r)$ of C_r

In this section you will explore the geometry of the image $f(C_r)$ in the w-plane for some polynomials $f(z)$. Consider the polynomial $g(z) = z^3 + z$.

Question 3 Change the program that you wrote for Question 1 so that it plots the image $g(C_r)$ for a given r. Examine what happens as r increases from a very small value to a moderately large one. In your write-up explain what happens. Use plots of $g(C_r)$ for suitably chosen values of r to illustrate your explanation.

Question 4 Repeat Question 3 for the polynomial $h(z) = z^2 + 2z + 1$. Again use plots of $h(C_r)$, along with figures chosen to zoom in on relevant details of the image curves.

Question 5 In the light of what you have found in Questions 3 and 4, explain what happens to the image curve $f_1(C_r)$ of the original polynomial f_1 in Question 1 as r increases from a suitably small value to a large one. Again, use plots of $f_1(C_r)$ for suitably chosen values of r to illustrate your explanation, including those chosen to zoom in on particular details.

4 Curvature of images $f(C_r)$ of C_r

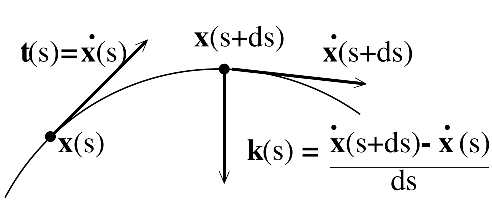

Figure 1: Coordinates and vectors associated with a curve

A smooth curve $\mathbf{x} : [a, b] \to \mathbb{C}$ is said to be *regular* if $\mathbf{x}'(t) \neq \mathbf{0}$ for all $t \in [a, b]$. It is a fact that any smooth regular curve $\mathbf{x}$ admits a smooth reparametrisation $\mathbf{x}(s)$ (where we are using $\mathbf{x}$ for both the original and reparametrised curve) such that the parameter s is the distance travelled along the curve, which we regard as the "natural" parametrisation. Figure 1 shows coordinates and vectors associated with a curve such as $f(C_r)$. Here $\mathbf{x}(s)$ is the position vector of a point on the curve. The distance from $\mathbf{x}(s)$ to $\mathbf{x}(s + ds)$ is $dx = |\mathbf{x}(s + ds) - \mathbf{x}(s)|$ for an infinitesimal distance ds along the curve. Hence $|\dot{\mathbf{x}}(s)| = 1$ where

$$\dot{\mathbf{x}}(s) = \lim_{ds \to 0} \frac{\mathbf{x}(s + ds) - \mathbf{x}(s)}{ds} \tag{2}$$

(this is sometimes phrased as $\mathbf{x}(s)$ is a unit speed curve). Hence the tangent vector $\mathbf{t}(s) = \dot{\mathbf{x}}(s)$ to the curve $\mathbf{x}(s)$ is always a unit vector. The vector $\mathbf{k}(s) = \ddot{\mathbf{x}}(s)$ is the curvature vector on the curve at the point $\mathbf{x}(s)$. The curvature $|\kappa|$ of the curve at the point $\mathbf{x}(s)$ is the magnitude of $\mathbf{k}(s)$,

$$|\kappa| = |\mathbf{k}(s)| \ , \tag{3}$$

and the radius of curvature ρ is

$$\rho = \frac{1}{|\kappa|} = \frac{1}{|\mathbf{k}(s)|} \ . \tag{4}$$

The function $f(z)$ is not a natural representation of the curve $f(C_r)$ because neither is z a scalar nor is $z_2 - z_1$ the distance in the complex plane along the curve between the two points $f(z_1)$ and $f(z_2)$. However, by using suitable coordinate transforms, an expression for the curvature vector can be found for an arbitrary parametric representation of $f(z)$.

To do so, write $f(z)$ in terms of the angle $\phi = \arg z$ to give a representation $\mathbf{x} = \mathbf{x}(\phi)$ of the curve $f(C_r)$ in terms of ϕ:

$$x(\phi) = \mathrm{Re}\,[f(z(\phi))] = \mathrm{Re}\,[f(\phi)] \tag{5}$$
$$y(\phi) = \mathrm{Im}\,[f(z(\phi))] = \mathrm{Im}\,[f(\phi)] \ , \tag{6}$$

where $\mathbf{x}(\phi) = (x(\phi), y(\phi))$.

Question 6 Show that

$$|\kappa| = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3} \ , \tag{7}$$

where

$$\mathbf{x}' = \left(\mathrm{Re}\left[\frac{df(z(\phi))}{d\phi}\right], \ \mathrm{Im}\left[\frac{df(z(\phi))}{d\phi}\right]\right) \tag{8}$$

$$\mathbf{x}'' = \left(\mathrm{Re}\left[\frac{d^2 f(z(\phi))}{d\phi^2}\right], \ \mathrm{Im}\left[\frac{d^2 f(z(\phi))}{d\phi^2}\right]\right) \ . \tag{9}$$

Equation (7) gives us the magnitude but not the sign of κ. We (arbitrarily) define the sign of κ to be positive if the curve $\mathbf{x}(s)$ is turning anticlockwise about its local centre of rotation. By expressing $\mathbf{x}'$ and $\mathbf{x}''$ in terms of $\mathbf{x}(s)$, $\mathbf{t}(s)$, $\mathbf{k}(s)$ and derivatives of s with respect to ϕ, find a way to compute the sign of κ.

Write (8) and (9) in terms of derivatives of $f(z)$ with respect to z.

Programming Task: Write a program to compute the integral

$$\kappa_{\mathrm{tot}} = \int_{f(C_r)} \kappa \, ds \tag{10}$$

for a range of values of r. Use your answer to Question 6 to ensure that your program calculates the sign of κ correctly.

Question 7 Using your program plot a graph of κ_{tot} against r for each of the polynomials $f_1(z)$, $g(z)$ and $h(z)$. Use what you have found out in Questions 1–6 to help you to explain what you see in your graphs. For a general polynomial $f(z)$, what does κ_{tot} tell you about the curve $f(C_r)$?

Reference

Lipschutz, M. M., 1969: *Schaum's Outline Series: Theory and Problems of Differential Geometry*, McGraw-Hill Education – Europe.

