2.2 Parallel-Plate Capacitor: Laplace's Equation

This project is self-contained. Part IB Electromagnetism provides background but is not necessary.

1 Introduction

You will solve Laplace's equation in two dimensions, in order to compute properties of a simple capacitor. The system is sketched in Fig. 1, it consists of two parallel rectangular plates of size $\ell_x \times \ell_z$, separated by a distance d. The thickness of the plates is negligible. Let the electric potential at position (x, y, z) be $\varphi(x, y, z)$. The centres of the plates are at $(x, y, z) = (0, \pm\frac{d}{2}, 0)$. On the upper plate then $\varphi = +V/2$ and on the lower plate then $\varphi = -V/2$.

The problem is to find a function $\phi(x, y)$ that solves

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1}$$

subject to boundary conditions:

$$\begin{aligned} &\phi = (\pm V/2) \text{ on the plates: } y = (\pm d/2) \text{ and } |x| < (\ell_x/2) \\ &\phi \to 0 \text{ as either } |x| \to \infty \text{ or } |y| \to \infty \end{aligned} \tag{2}$$

Assume that $\ell_z \gg \ell_x$: then the solution $\phi(x, y)$ is an accurate approximation to $\varphi(x, y, 0)$, which is the potential in the plane $z = 0$.

The electric field (in this plane) is given by

$$\mathbf{E} = -\nabla\phi = \left(-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y} \right) . \tag{3}$$

It is useful to work in dimensionless units, defined as follows:

$$X = \frac{2x}{d} , \qquad Y = \frac{2y}{d} , \qquad L = \frac{\ell_x}{d} , \qquad \Phi = \frac{\phi}{V} , \tag{4}$$

where q is computed for the upper plate. This simplifies our problem to

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} = 0 , \tag{5}$$

with $\Phi = \pm\frac{1}{2}$ on the plates, which are at $Y = \pm 1$ with $|X| \leqslant L$; also $\Phi \to 0$ far from the plates.

For large L, the expected physical behaviour *in the gap between the plates* is that $\nabla\Phi$ is (almost) aligned with the Y-direction, and depends weakly on X, Y.

2 Numerical Method

For numerical purposes, it is convenient to solve Eq. (5) in a rectangular domain $\mathcal{D} = [-D_X, D_X] \times [-D_Y, D_Y]$ where D_X, D_Y are parameters to be chosen appropriately. We set $\Phi = 0$ on the boundary of this domain*: the solution to the original problem is recovered by taking $D_X, D_Y \to$

*This corresponds physically to putting the capacitor in a conducting box at potential $\phi = 0$.

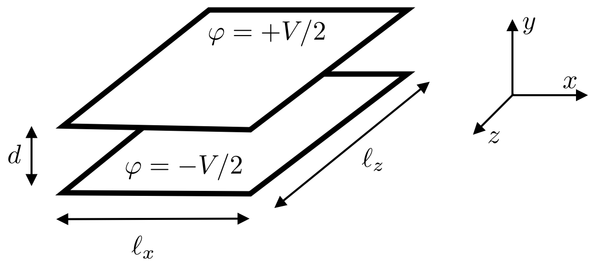

Figure 1: Sketch of two parallel plates of size $\ell_x \times \ell_z$, separated by distance d. The upper plate is at potential $\varphi = V/2$ and the lower one is at $\varphi = -V/2$.

∞. The solution Φ has symmetries $\Phi(-X, Y) = \Phi(X, Y)$ and $\Phi(X, -Y) = -\Phi(X, Y)$. Hence it is sufficient to compute Φ in the positive quadrant of $\mathcal{D}$.

Within the domain $\mathcal{D}$, define a grid of points $(X_m, Y_n) = (mh, nh)$ where m, n are integers and h is the grid spacing. The domain $\mathcal{D}$ should be chosen such that $N_x = D_X/h$ and $N_y = D_Y/h$ are integers, as is $1/h$. You will compute a numerical approximation to $\Phi(X_m, Y_n)$ which is denoted by $\Phi_{m,n}$.

> **Question 1** Suppose that we can find a numerical solution [for all grid points (X_i, Y_j)] of the equation
>
> $$\Phi_{i-1,j} + \Phi_{i+1,j} + \Phi_{i,j-1} + \Phi_{i,j+1} - 4\Phi_{i,j} \;=\; 0 \tag{6}$$
>
> subject to appropriate boundary conditions. Show that as $h \to 0$, this numerical solution approximates a solution of the Laplace equation (5).

Eq. 6 can be solved by iteration. Starting with an initial guess $\Phi^{(0)}$ one defines a sequence of approximations $\Phi^{(k)}$ with $k = 1, 2, \ldots$, which converge to the solutions of Eq. (6). For some mesh points, the value of $\Phi_{m,n}$ is fixed by boundary conditions (either from the plates or from the boundary of $\mathcal{D}$). For the remaining points, an iteration rule is required.

A simple rule is the Jacobi scheme

$$\Phi^{(k+1)}_{i,j} \;=\; \frac{1}{4}\left[\Phi^{(k)}_{i-1,j} + \Phi^{(k)}_{i+1,j} + \Phi^{(k)}_{i,j-1} + \Phi^{(k)}_{i,j+1}\right] \tag{7}$$

However, it is more efficient in practice to use the successive over-relaxation (SOR) method:

$$\Phi^{(k+1)}_{i,j} = (1-\omega)\Phi^{(k)}_{i,j} \;+\; \frac{\omega}{4}\left[\Phi^{(k+1)}_{i-1,j} + \Phi^{(k)}_{i+1,j} + \Phi^{(k+1)}_{i,j-1} + \Phi^{(k)}_{i,j+1}\right] \tag{8}$$

where ω is a parameter with $1 \leqslant \omega < 2$. The case $\omega = 1$ is called Gauss-Seidel iteration, larger ω corresponds to increasing "over-relaxation" which can be effective for accelerating convergence. Note that the right hand side of Eq. 8 mixes quantities from the kth and $(k+1)$th iterations, this is feasible in practice because the $\Phi^{(k+1)}_{m,n}$ are computed sequentially in m, n. Full details can be found in texts on numerical methods, such as Ref. [1].

3 Computing the potential and the electric field

When implementing the SOR method, you should restrict to the positive quadrant of $\mathcal{D}$, and you will need to take care with the iteration rule at the edge of this domain. You can use that $\Phi = 0$ on the boundaries $X = D_X$ and $Y = D_Y$. For $Y = 0$ then $\Phi = 0$ since Φ is odd in Y. For $X = 0$ then you should replace $\Phi^{(k+1)}_{-1,j}$ in the iteration rule by $\Phi^{(k)}_{1,j}$, using that $\Phi(-h, Y) = \Phi(h, Y)$ by symmetry.

Note carefully that the boundary conditions fix Φ for points on the plates, so the iteration rule should not be applied there.

You will also need a criterion for stopping the iteration. For this, define the residual

$$r_k = \frac{1}{N} \sum_i \sum_j \left| \Phi^{(k)}_{i,j} - \Phi^{(k-1)}_{i,j} \right| \tag{9}$$

where N is the total number of mesh points, and the sum runs over all such points. The kth iteration $\Phi^{(k)}$ can be taken as a suitable approximation for the solution Φ if $r_k < \epsilon_{\text{tol}}$, where ϵ_{tol} is a small tolerance parameter that you will need to choose.

Programming Task: Write a program to implement the SOR iteration method. The program output will depend on parameters $L, D_X, D_Y, h, \omega, \epsilon_{\text{tol}}$. It will be necessary to plot the solution Φ, either as a function of two variables, or as one-dimensional "slices" along the x or y directions. The validity of the method does not depend on your initial guess $\Phi^{(0)}$, you should verify this.

Question 2 Test your program as follows. Take $L = 1$ and $h = \frac{1}{2}$ and $(D_X, D_Y) = (2, 2)$. Take $\omega = 1$. By suitably adjusting ϵ_{tol}, verify that the solution to the discretised problem in Eq. (6) has $\Phi_{1,1} = \Phi_{1,3} = 0.238$ (to three significant figures). Show how your estimate of Φ depends on (X, Y) inside $\mathcal{D}$.

The Y-component of the electric field (in dimensionless units) is $\mathcal{E}_Y = -\partial\Phi/\partial Y$, which can be estimated by a finite difference as

$$\mathcal{E}_Y(X, Y) \approx \pm\frac{1}{h}[\Phi(X, Y \mp h) - \Phi(X, Y)] \tag{10}$$

Note that $\mathcal{E}_Y$ may be discontinuous at $Y = 1$, in which case the left- and right-derivatives are not equal (but both can be estimated by choosing appropriately the $\pm$ signs).

Question 3 Take $L = 2$ and $h = \frac{1}{4}$ and $(D_X, D_Y) = (4, 4)$. Plot the corresponding numerical approximations to $\Phi(0, Y)$ and $\Phi(2, Y)$ for $0 < Y < D_Y$. Plot estimates of the field on the mid-plane: $\mathcal{E}_Y(X, 0)$ for $0 < X < D_X$. Also plot the electric field on the upper and lower surfaces of the plate $\mathcal{E}_Y(X, Y \to 1)$. Comment on how you ensured that your results solve Equ. (6) to sufficient accuracy.

Question 4 For the parameters of question 3, investigate the effect of reducing h (for example, you might compare $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{12}$). Plot the same quantities considered in that question, for several different values of h, on the same axes. Are your results consistent with convergence to a suitable solution of the Laplace equation, as $h \to 0$?

Question 5 Still for the parameters of question 3, how many iterations of SOR are required for convergence? Investigate how this depends on ω.

Question 6 Recall that the original problem of interest was posed on the infinite domain $X, Y \in \mathbb{R}^2$ with boundary condition $\Phi \to 0$ as $|X|, |Y| \to \infty$. By varying D_X and D_Y, investigate how much your numerical solutions near the plates are affected by the the boundary condition that $\Phi = 0$ on the boundary of $\mathcal{D}$.

4 Comparison with semi-infinite plates

To understand the behavior of Φ near the ends of the plates, it is useful to consider a slightly different problem, for which exact results are available. Instead of two finite plates, we consider semi-infinite ones corresponding to the line segments $Y = \pm 1$ with $X \in (-\infty, L]$. In this case, the theory of conformal mappings provides formulae for *equipotentials* (lines of constant Φ).†

The theory defines a function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ such that lines of constant Ψ are everywhere orthogonal to the equipotentials.‡ Now define $W = -\Phi + \mathrm{i}\Psi$ (where $\mathrm{i} = \sqrt{-1}$, as usual). Then

$$(X - L) + \mathrm{i}Y = \frac{1 + \mathrm{e}^{-2\pi \mathrm{i} W}}{\pi} - 2\mathrm{i}W \ . \tag{11}$$

Equipotential lines can be computed (parametrically) from this formula by fixing some $\Phi \in [-\frac{1}{2}, \frac{1}{2}]$, varying Ψ, and taking real/imaginary parts. Lines of constant Ψ (known as *field lines*) are obtained similarly, by fixing Ψ and varying Φ. Obviously, the value of L only shifts these solutions along the X-direction.

In addition, the electric field $(\mathcal{E}_X, \mathcal{E}_Y) = (-\partial\Phi/\partial X, -\partial\Phi/\partial Y)$ can be obtained in terms of Φ, Ψ from the following formula:

$$-\frac{\partial \Phi}{\partial X} + \mathrm{i}\frac{\partial \Phi}{\partial Y} = \frac{\mathrm{i}}{2(\mathrm{e}^{-2\pi \mathrm{i} W} + 1)} \tag{12}$$

Question 7 Consider the semi-infinite case with $L = 0$ and plot some illustrative equipotentials and field lines. (When plotting equipotentials, some care is required with the range of Ψ-values.) The upper surface of the top plate corresponds to $\Phi = \frac{1}{2}$ and $\Psi \in (0, \infty)$ while the lower surface is $\Phi = \frac{1}{2}$ and $\Psi \in (-\infty, 0)$: illustrate this by plotting some equipotentials with $\Phi = \frac{1}{2} - \delta$, for suitably small δ.

Question 8 Consider the electric field on the upper surface of the top plate, as follows: You know from Question 7 how Ψ and Φ behave on the plate. Use Eq. (12) to show that the electric field on the surface is in the Y direction and derive its magnitude in terms of Ψ. Consider the asymptotic behaviour of Eqs. (11,12) as $\Psi \to 0^+$ and $\Psi \to \infty$ (always with $\Phi = \frac{1}{2}$) and hence show that

$$\mathcal{E}_Y(X, 1^+) \approx \begin{cases} a(L - X)^{-1/2} & \text{as } X \to L^- \\ b(L - X)^{-1} & \text{as } X \to -\infty \end{cases}$$

where a and b are constants to be determined.

What happens on the lower surface of the plate?

Question 9 For large values of L, D_X, D_Y, the numerical solutions for finite plates can be compared with the semi-infinite case. For a few values of L, make plots that compare your numerical estimates of the field on the plate surfaces with the results for semi-infinite plates derived in Question 8, which can again be plotted parametrically (by varying Ψ).

For the numerical solutions, you will need to choose values of h, D_X, D_Y that balance the computational time with the accuracy required. In which parts of the domain do the numerical solutions depend most strongly on these parameters (and on L)? In which parts of the domain do the solutions match for finite and semi-infinite cases?

†A detailed discussion is given in Ref. [2], including some results for finite plates, but these are not needed for this project.

‡This Ψ is a harmonic conjugate of $-\Phi$, which means that $\partial\Phi/\partial X = -\partial\Psi/\partial Y$ and $\partial\Phi/\partial Y = \partial\Psi/\partial X$.

References

1. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, CUP, 1996.

2. H. B. Palmer, The Capacitance of a Parallel-Plate Capacitor by the Schwartz-Christoffel Transformation, *Transactions of the American Institute of Electrical Engineers*, vol.56(3), pp.363-366 (1937). See http://dx.doi.org/10.1109/T-AIEE.1937.5057547.