2.1 The Restricted Three-Body Problem

Knowledge of the Part IA Dynamics and Relativity lecture course is useful for this project.

1 Introduction

Determining the motion of gravitating bodies is a classical problem. For two bodies there is a stable analytic solution which describes rotation about the joint centre of mass. For three bodies, the problem cannot be solved analytically. Various simplifications have historically been considered, one of which is the 'restricted three-body problem' in which the third body is taken to be much smaller in mass than the other two and therefore does not affect their motion. The problem is then to solve for the motion of the third body, under the gravitational influence of the other two.

It is convenient to transform to a rotating frame of reference in which the first two bodies appear stationary and the origin corresponds to their joint centre of mass. Scalings may be chosen so that the angular velocity of this frame is 1 and the distance between the two bodies is 1. The only parameter then appearing is the quantity μ defined such that the two masses are in the ratio $\mu : 1-\mu$ and are situated respectively at the points $(\mu-1, 0)$ and $(\mu, 0)$. These points are denoted by P_1 and P_2. At time t, the third body has position $(x(t), y(t))$. Its equation of motion is

$$\ddot{x} - 2\dot{y} = -\frac{\partial\Omega}{\partial x}\,, \tag{1}$$

$$\ddot{y} + 2\dot{x} = -\frac{\partial\Omega}{\partial y}\,, \tag{2}$$

where

$$\Omega = -\tfrac{1}{2}\mu r_1^2 - \tfrac{1}{2}(1-\mu)r_2^2 - \frac{\mu}{r_1} - \frac{1-\mu}{r_2}\,, \tag{3}$$

with $r_1^2 = (x+1-\mu)^2 + y^2$ and $r_2^2 = (x-\mu)^2 + y^2$.

Despite the substantial restriction to the full three-body problem which this represents, it is not possible to solve (1) and (2) analytically.

Question 1 Show from (1) and (2) that the quantity

$$J = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 + \Omega(x, y)$$

is constant following the motion. Deduce that trajectories must be confined to the region

$$\Omega(x, y) \leqslant \Omega(x_0, y_0) + \tfrac{1}{2}u_0^2 + \tfrac{1}{2}v_0^2\,, \tag{4}$$

where x_0, y_0, u_0 and v_0 are the initial values of x, y, $\dot{x}$ and $\dot{y}$, respectively.

Programming Task: Write a program to solve (1) and (2) numerically, given suitable initial conditions on x, y, $\dot{x}$ and $\dot{y}$. You may wish to use a suitable MATLAB ODE solver, such as `ode45`, which automatically adjust the time-step. You may need to adjust the error tolerance of your solver. If you use a different programming language and a fixed-step integration routine, you may need to use very small time-steps.

Write a second program to plot contours of $\Omega(x, y)$ in the (x, y) plane.

Throughout this project, it will be necessary to check that your program is yielding accurate results. Standard checks include (i) testing the program against known analytic solutions (if there are any), and (ii) varying the time-step or error tolerance. The fact that J is constant provides another possible check on the accuracy of the numerical solution. In your report, comment on the checks you have performed.

2 Space travel

Assume that the third body is a spacecraft, with the first two bodies being planets of equal mass ($\mu = 0.5$).

Question 2 Consider motion in the neighbourhood of P_2, so that the effect of P_1 may be ignored and (3) may be approximated by

$$\Omega = -\frac{1}{2r_2} \ . \tag{5}$$

Show that (1) and (2) then have analytic solutions with the spacecraft in a circular orbit of radius a about P_2, where a can take any value. [The approximation (5) is of course only valid if a is small.]

Modify your program to solve (1) and (2) with Ω specified by (5) instead of (3). Demonstrate (for one value of a) that the modified program can reproduce the analytic solutions.

Question 3 Now return to the original problem (1)–(3) with $\mu = 0.5$, and take initial conditions $x = 0.2$, $y = 0$, $\dot{x} = 0$, $\dot{y} = v_0$ with $v_0 = -0.50$, -1.00, -1.04, -1.18, and -1.25, -1.50 in turn. Integrate from $t = 0$ to $t = 15$. For each case, make a plot that shows the trajectory, together with the allowed region $\Omega(x, y) \leqslant J$ (or its complement). Give an indication of the accuracy of your numerical integrations.

Comment on the trajectories, and how these and the allowed region change as v_0 increases. Is the allowed region a useful guide to the size of the trajectory? How does a trajectory behave close to the boundary of the allowed region? Which value of v_0 would be most suitable to travel from the neighbourhood of P_2 to the neighbourhood of P_1?

3 Lagrange points and asteroids

In this part of the project do not restrict attention to the case $\mu = 0.5$.

Question 4 By examining contour plots of Ω show that the system (1)–(3) generally has five equilibrium points. (Three lie on the x-axis and the other two — the *equilateral Lagrange points* — at the third vertex of an equilateral triangle whose other two vertices are at P_1 and P_2.) Display contour plots for three values of μ, with the equilibrium points marked.

Investigate the stability of the different equilibrium points numerically, i.e., by starting trajectories a small distance away from the equilibrium point and integrating forward in time. Display plots of some representative trajectories in the (x, y) plane to illustrate your results. (You may also find it useful to show plots of x or y against t.)

What do you conclude about the stability of the equilibrium points on the x-axis? Do the stability properties depend on μ? Confirm your findings by performing a linearised stability analysis about such points. You should be able to deduce any necessary information about the second derivatives of Ω by considering the shapes of the contours, rather than by detailed calculation.

Question 5 Continue with a numerical investigation of the stability of the equilateral Lagrange points, for parameter values $\mu = 0.008$, 0.022, 0.044, 0.08 and 0.5. Illustrate the results in your write-up with one trajectory picture for each.

How do the stability properties change with μ? Either by numerical experimentation, or by performing a linearised stability analysis, find (to two significant figures) the critical value μ_c dividing values of μ for which the point is stable from those for which it is unstable. [This time the linearised stability analysis does require calculation of the second derivatives of Ω.] For the stable cases give a qualitative description of the motion.

Question 6 The Trojans are a group of asteroids observed at the Sun–Jupiter equilateral Lagrange point. For the Sun-Jupiter system $\mu = 9.54 \times 10^{-4}$. Is the persistence of the Trojans at this point consistent with your findings above? The Earth–Moon system has $\mu = 0.012141$. Would you expect analogues of the Trojans at this point?

