1.2 Ordinary Differential Equations

This project builds on material covered in the Part IA Differential Equations and the Part IA lectures on Computational Projects.

1 Numerical solution of first order ODEs

The objective of this section is the numerical step-by-step integration of ordinary differential equations (ODEs) of the form

$$\frac{dy}{dx} = f(x, y)\,. \tag{1a}$$

An initial condition is specified at, say, $x = x_0$, i.e.

$$y\,(x_0) = Y_0\,. \tag{1b}$$

The numerical methods to be investigated are as follows:

(a) The **Euler** (or more precisely **forward Euler**) method employs the scheme

$$Y_{n+1} = Y_n + hf(x_n, Y_n)\,, \tag{2}$$

where Y_n denotes the numerical solution at $x_n \equiv x_0 + nh$, that is, the solution at the nth step with step length h, starting from x_0 and Y_0.

Definition. The *global error* after the nth step is defined as

$$E_n = Y_n - y(x_n)\,, \tag{3a}$$

where $y(x_n)$ is the exact solution to (1a) at x_n.

Definition. The *local error* of the first step is defined as $e_1 = Y_1 - y(x_1)$. For subsequent steps the *local error* is defined as

$$e_n = Y_n - w(x_n)\,, \tag{3b}$$

where $w(x_n)$ is the exact solution to (1a) at $x = x_n$ starting from $x = x_{n-1}$ and $y = Y_{n-1}$. Note that, in general, $Y_{n-1} \neq y(x_{n-1})$ for $n > 1$.

For the Euler method it can be shown that e_n is $O(h^2)$ as $h \to 0$. As a result the Euler method is said to have *first-order accuracy.*

(b) The fourth-order **Runge–Kutta** (RK4) method employs the scheme:

$$Y_{n+1} = Y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\,, \tag{4a}$$

where

$$k_1 = hf(x_n,\ Y_n)\,, \tag{4b}$$
$$k_2 = hf(x_n + \tfrac{1}{2}h,\ Y_n + \tfrac{1}{2}k_1)\,, \tag{4c}$$
$$k_3 = hf(x_n + \tfrac{1}{2}h,\ Y_n + \tfrac{1}{2}k_2)\,, \tag{4d}$$
$$k_4 = hf(x_n + h,\ Y_n + k_3)\,. \tag{4e}$$

The RK4 method has *fourth-order accuracy*, i.e. e_n is $O(h^5)$ as $h \to 0$.

Theoretical background for the stability and accuracy of these methods can be found in, for example, *An Introduction to Numerical Methods and Analysis* by J.F.Epperson, *An Introduction to Numerical Methods* by A.Kharab and R.B.Guenther and *Numerical Recipes* by Press *et al.*

2 Stability and accuracy of the numerical methods

This section will explore numerical integration of equation (1a) with

$$f(x, y) = -16y + 15\mathrm{e}^{-x} \tag{5a}$$

and initial condition

$$y(0) = 0 \; . \tag{5b}$$

This has the exact solution

$$y(x) = \mathrm{e}^{-x} - \mathrm{e}^{-16x} \; . \tag{6}$$

Programming Task: Write program(s) to apply the Euler and RK4 methods to this problem.

2.1 Stability

Question 1 Using the Euler method, starting with $Y_0 = 0$, compute Y_n for x up to $x = 6$ with $h = 0.6$, i.e. for n up to $6/h = 10$. Tabulate the values of x_n, the numerical solution Y_n, the analytic solution $y(x_n)$ from (6), and the global error $E_n \equiv Y_n - y(x_n)$. You should find that the numerical result is unstable: the error oscillates with a magnitude that ultimately grows proportional to $\mathrm{e}^{\gamma x}$, where the 'growth rate' γ is a positive constant which you should estimate.

Repeat with $h = 0.4$, 0.2, 0.125 and 0.1, presenting only a judicious selection of output to illustrate the behaviour. What effect does reducing h have on the size of the instability, and on its growth rate?

Question 2
(i) Find the analytic solution of the Euler *difference* equation

$$Y_{n+1} = Y_n + h\left(-16Y_n + 15\left(\mathrm{e}^{-h}\right)^n\right) \quad \text{with} \quad Y_0 = 0\,. \tag{7}$$

(ii) Hence explain why and when instability occurs, and with what growth rate.

(iii) Show that in the limit $h \to 0$, $n \to \infty$ with $x_n \equiv nh$ fixed, the solution of the difference equation (7) converges to the solution (6) of the differential equation specified by (1a), (5a) and (5b).

2.2 Accuracy

Question 3 Integrate the ODE specified by specified by (1a), (5a) and (5b) numerically with $h = 0.05$ from $x = 0$ to $x = 4$ using both the Euler and RK4 methods. Plot Y_n against x_n for each method with the exact solution (6) superposed.

Question 4 For both the Euler and the RK4 methods, tabulate the global error E_n at $x_n = 0.1$ against $h \equiv 0.1/n$ for $n = 2^k$ with $k = 0, 1, 2, \ldots, 15$, and plot a log–log graph of $|E_n|$ against h over this range.

Comment on the relationship of your results to the theoretical accuracy of the methods.

3 Numerical solutions of second-order ODEs

The same time-stepping methods can also be applied to higher order ODEs. This section will explore solutions to a damped harmonic oscillator with a driving force. Specifically, start by considering the following equation

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + y = \sin(\omega t), \tag{8}$$

where γ and ω are non-negative real constants and t and y are real variables.

Question 5 Find the general solution of (8) for the lightly damped case, $0 < \gamma < 2$. Show that

$$y \to A_s \sin(\omega t - \phi_s) \quad \text{as } t \to \infty \tag{9}$$

where the 'steady-state' amplitude A_s and 'steady-state' phase shift ϕ_s are to be found in terms of γ and ω.

Equation (8) can be rewritten as a pair of coupled first-order ODEs for

$$y^{(1)}(t) \equiv y(t) \qquad \text{and} \qquad y^{(2)}(t) \equiv \frac{dy(t)}{dt}\ , \tag{10}$$

namely

$$\frac{dy^{(1)}}{dt} = f^{(1)}(t, y^{(1)}, y^{(2)}) \equiv y^{(2)}\ , \tag{11}$$

$$\frac{dy^{(2)}}{dt} = f^{(2)}(t, y^{(1)}, y^{(2)}) \equiv -\gamma y^{(2)} - y^{(1)} + \sin(\omega t). \tag{12}$$

This system of equations can be solved using either the Euler or the RK4 method, but here we will just use the latter.

The RK4 method can be generalised to solve first order systems of equations by writing

$$\mathbf{Y}_{n+1} = \mathbf{Y}_n + \tfrac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)\,, \tag{13a}$$

where

$$\mathbf{k}_1 = h\mathbf{f}(x_n,\ \mathbf{Y}_n)\,, \tag{13b}$$

$$\mathbf{k}_2 = h\mathbf{f}(x_n + \tfrac{1}{2}h,\ \mathbf{Y}_n + \tfrac{1}{2}\mathbf{k}_1)\,, \tag{13c}$$

$$\mathbf{k}_3 = h\mathbf{f}(x_n + \tfrac{1}{2}h,\ \mathbf{Y}_n + \tfrac{1}{2}\mathbf{k}_2)\,, \tag{13d}$$

$$\mathbf{k}_4 = h\mathbf{f}(x_n + h,\ \mathbf{Y}_n + \mathbf{k}_3) \tag{13e}$$

where $\mathbf{f}$, $\mathbf{Y}$, and $\mathbf{k}$ are two-dimensional vectors and the components of the vectors $\mathbf{f}$ and $\mathbf{Y}$ can be written $\mathbf{f} = (f^{(1)}, f^{(2)})$ and $\mathbf{Y} = (Y^{(1)}, Y^{(2)})$.

Programming Task: Write a program to solve equation (8) using the RK4 method, subject to the initial conditions

$$y = \frac{dy}{dt} = 0 \quad \text{at } t = 0\ . \tag{14}$$

Question 6 Taking $\gamma = 1$ and $\omega = \sqrt{3}$, use your program to compute Y_n for t up to 10 with $h = 0.4$ [i.e. for n up to 25], and tabulate the numerical solution Y_n, the analytic solution $y(t_n)$ and the global error $E_n \equiv Y_n - y(t_n)$ against t_n. Repeat with both $h = 0.2$ and $h = 0.1$ [integrating up to $t = 10$, i.e. for n up to 50 and 100 respectively], not necessarily presenting all the output. Comment on the errors.

Question 7 Use your RK4 program (with suitable h) to generate and plot numerical solutions of (8) and (14) up to $t = 40$ for $\omega = 1$ and $\gamma = 0.25$, 0.5, 1.0 and 1.9, checking that they agree with the analytic solutions. Do likewise for $\omega = 2$ and the same values of γ. Explain the differences between the various cases in terms of the mathematics and the physics of the system under investigation.

The last question consider a case with nonlinear damping,

$$\frac{d^2y}{dt^2} + \frac{d}{dt}\left(\tfrac{1}{3}\delta^3 y^3\right) + y = \sin t \ , \tag{15}$$

for which an analytic solution is not available. The initial conditions are as before,

$$y = \frac{dy}{dt} = 0 \quad \text{at } t = 0 \ . \tag{16}$$

Question 8 For $\delta = 0.25$, 0.5, 1.0 and 20, use your RK4 program to generate and plot numerical solutions to (15)–(16) for t up to 60, using suitable value(s) of h (justify your choice). Comment on the solutions, comparing them with each other and with those of Question 7 for $\omega = 1$.

Hint: it *may* be helpful to observe that when δ is 'small', equation (15) has a 2π-periodic solution of the form

$$y = \sum_{n=-1}^{\infty} \delta^n y_n(t) \tag{17}$$

where each $y_n(t)$ is periodic in t with period 2π and

$$y_{-1}(t) = A\cos t \ , \quad y_0(t) = B\sin t + C\sin 3t \tag{18}$$

for suitable values of the constants A, B and C [recall that $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$, and note that to determine y_0 completely it is necessary to consider terms of order δ]. What if δ is 'large'?

Additional Reference

Boyce, W. E., and DiPrima, R. C., 2001, *Elementary Differential Equations and Boundary Value Problems*, 7th edition. Publ. John Wiley & Sons Inc.

Project 1.2: Ordinary Differential Equations

Marking Scheme and additional comments for the Project Report

The purpose of these additional comments is to provide guidance on the structure and length of your CATAM report. Use the same concepts to write the rest of the reports. To help you assess where marks have been lost, this marking scheme will be completed and returned to you during Lent Term. You are advised to keep a copy of your write-up in order to correlate your answers to the marks awarded.

Question no.	**Marks available**[1]	**Marks awarded**[2]
Programming task *Program:* for instructions regarding printouts and what needs to be in the write-up, refer to the introduction to the manual.		
Question 1 *Tables:* for presentation and layout, refer to the introduction. [*quarter page*][3]	C2	
Question 2 *Analytic solution:* do not include trivial steps in your working. [*approx. 15 lines*][3]	M2	
Question 3 *Graphs:* you may use one graph or two.	C1	
Question 4 *Graphs:* ditto. *Comments:* what can be said about how the global error E_n for each method varies with h? How is this reflected in the plots? [*approx. 3 lines*][3]	C1 M1	
Question 5 *Analytic solution:* do not include trivial steps in your working; be sure to state the results unambiguously. [*approx. 5 lines*][3]	M1	
Question 6 *Analytic and numerical solutions compared:* the purpose of this step is to check that the program works and gives accurate answers ('validation'). Do the errors behave as expected when h is decreased? [*quarter page*][3]	C2+M2	
Question 7 *Comments:* first identify the salient features of the plots. Examine the nature of the functions that you are plotting: what are their components and how do these contribute to the overall solutions? Then use mathematical arguments (cf. the Part IA course *Differential Equations*) to explain the behaviour of the plots; link to the theory of the physical system under investigation. [*one page*][3]	C1+M2	
Question 8 *Numerical solutions:* explain why you are satisfied that your chosen value(s) of h will deliver sufficiently accurate results. *Comments:* identify the key similarities and differences between the various solutions, and with the help of the hint, or otherwise, try to explain them mathematically and/or physically. [*one page*][3]	C1 M2	
Excellence marks awarded for, *among other things*, mathematical clarity and good, clear output (graphs and tables) — see the introduction to the project manual.	E2	
Total Raw Marks	20	
Total Tripos Marks	**40**	

[1] C#, M# and E#: *Computational*, *Mathematical* and *Excellence* marks respectively.
[2] For use by the assessor.
[3] Your aim is to answer succinctly the questions including graphs and tables, and to make all important points. The length specified here should b sufficient for you to do this but is not a target.

