Open Day 2021: STEP workshops questions

The STEP workshops on our virtual Open Day are designed to guide you through a few STEP questions from past STEP paper.

You are very much encouraged to have a look at the three questions below, and try to have a go at solving them **before** you watch the workshops.

Don't expect to find it easy! It'll probably look very difficult and very different from the sort of maths questions you are used to, especially if you have never looked at a STEP question before, and you may not be able to make much progres with it. Nevertheless, if you start thinking about it, then you will get much more out of watching videos of the STEP workshops.

The question below was **Q2 from STEP paper 2 in 2016**, and was explained in workshop 1.

Q1. Use the factor theorem to show that $a+b-c$ is a factor of

$$(a+b+c)^3 - 6(a+b+c)(a^2+b^2+c^2) + 8(a^3+b^3+c^3)\,. \qquad (*)$$

Hence factorise $(*)$ completely.

(i) Use the result above to solve the equation

$$(x+1)^3 - 3(x+1)(2x^2+5) + 2(4x^3+13) = 0\,.$$

(ii) By setting $d+e=c$, or otherwise, show that $(a+b-d-e)$ is a factor of

$$(a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3)$$

and factorise this expression completely.

Hence solve the equation

$$(x+6)^3 - 6(x+6)(x^2+14) + 8(x^3+36) = 0\,.$$

(See next page for some advice.)

This question is fairly typical of many STEP questions, by having a 'stem' (a kind of introductory part in which mathematical ideas and results are stated and/or derived), followed by two other parts where the ideas and results from the 'stem' are used to find the answers to questions. Therefore, you should read the 'stem' carefully, and use the ideas and results obtained to find the solution to the rest of the question.

Even though there are usually several different ways of obtaining solutions to a mathematical question, it is always wise to use the directions and hints given. These are likely to make the solution easier, and are best used also to get full marks, unless the direction explicitly gives you a choice.

The first direction is: "Use the factor theorem", so it really is a good idea to use it! Candidates who attempted to factorise (*) without using the factor theorem ended up with some very complicated and messy algebraic expressions that were difficult to simplify, so more prone to mistakes, even though it's possible to obtain the answer.

If you find this question in your exam and you've never heard about the factor theorem, it may be best to skip it. In the examination, there is a choice of questions (you only need to choose up to 6 questions), so those candidates who don't feel comfortable with the question can simply do a different one.

You should also note - and follow! - the subsequent directions:

(i) Use the result above ... [definitely use]

(ii) By setting d+e =c or otherwise ... [only a suggestion, but very likely to make your life easier if you follow it]

The question below was **Q2 from STEP paper 2 in 2012**, and was explained in workshop 2.

Q2. If $\mathrm{p}(x)$ and $\mathrm{q}(x)$ are polynomials of degree m and n, respectively, what is the degree of $\mathrm{p}(\mathrm{q}(x))$?

(i) The polynomial $\mathrm{p}(x)$ satisfies

$$\mathrm{p}(\mathrm{p}(\mathrm{p}(x))) - 3\mathrm{p}(x) = -2x$$

for all x. Explain carefully why $\mathrm{p}(x)$ must be of degree 1, and find all polynomials that satisfy this equation.

(ii) Find all polynomials that satisfy

$$2\mathrm{p}(\mathrm{p}(x)) + 3[\mathrm{p}(x)]^2 - 4\mathrm{p}(x) = x^4$$

for all x.

(See next page for some advice.)

Here's another example where what is asked in the 'stem' helps you find the answers to the other parts of the question.

Don't be put off by the expression $p(q(x))$ (or $p(p(p(x)))$!), where the variable in a polynomial is itself another polynomial. Just write out the expression for a generic polynomial $p(x)$ of degree n (no need to write out all terms: only those which are important for the reasoning), then think what happens if in each term you substitute x with another generic polynomial (again, most definitly no need to write out all terms!).

Then, when answering part (i), you must be rigorous and systematic: you can't just show that if the degree of $p(x)$ is 1 then the expression given is satisfied - you must also show that all other possible values for the degree of $p(x)$ (higher or smaller than 1) won't work. Note that you're asked to find **all** *polynomials that satisfiy the expression given. Be equally careful and systematic when answering part (ii).*

The question below was **Q1 from STEP paper 2 in 2011**, and was explained in workshop 3.

Q3. **(i)** Sketch the curve $y = \sqrt{1-x} + \sqrt{3+x}$.

Use your sketch to show that only one real value of x satisfies

$$\sqrt{1-x} + \sqrt{3+x} = x + 1\,,$$

and give this value.

(ii) Determine graphically the number of real values of x that satisfy

$$2\sqrt{1-x} = \sqrt{3+x} + \sqrt{3-x}\,.$$

Solve this equation.

(See next page for some advice.)

Curve sketching which involves visualising functions without the use of a graphical calculator, is considered to be a good test of mathematical understanding and crops up frequently in STEP.

You may be used to graphic calculators such as e.g. Desmos, and might wonder what the point is of doing the sketch without a calculator when it could be obtained much more easily with a calculator. The answer is that the final sketch itself is not particularly interesting or useful (except for the rest of the question) : who cares what it looks like? The interesting and useful part is the thought process that goes into working out how the sketch looks. If you use a calculator, the benefit of creating the sketch is lost to you for ever.

Often, as in the question above, the sketch has a purpose: part (i) asks you explicitly to "use your sketch" to show what is asked; part (ii) asks you explicitly to "determine graphically" the number of real values satisfying the given expression.

Since square roots appear in the function you need to sketch, you should be mindful that you're looking to plot real values (Cartesian plane), so you should think about the domain of your function. Apart from that, the sketches required for this question are not particularly tricky.