Open Day 2020: STEP workshops sample questions

The STEP workshops on our virtual Open Day are designed to guide you through a few STEP questions from past STEP paper.

You are very much encouraged to have a look at the four questions below, and try to have a go at solving them ***before*** you watch the workshops.

Don't expect to find it easy! It'll probably look very difficult and very different from the sort of maths questions you are used to, especially if you have never looked at a STEP question before. You may not be able to get anywhere with it, and we certainly do not expect you to be able to write a full solution before Saturday, but if you start thinking about it, then you will get much more out of watching videos of the STEP workshops.

The question below was **Q2 from STEP paper 1 in 2013**, and was explained in all workshops.

Q1. In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that $\lfloor 2.9 \rfloor = 2 = \lfloor 2.0 \rfloor$ and $\lfloor -1.5 \rfloor = -2$.

The function f is defined, for $x \neq 0$, by $\mathrm{f}(x) = \dfrac{\lfloor x \rfloor}{x}$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \leqslant x \leqslant 3$ (with $x \neq 0$).

(ii) By considering the line $y = \frac{7}{12}$ on your graph, or otherwise, solve the equation $\mathrm{f}(x) = \frac{7}{12}$.

Solve also the equations $\mathrm{f}(x) = \frac{17}{24}$ and $\mathrm{f}(x) = \frac{4}{3}$.

(iii) Find the largest root of the equation $\mathrm{f}(x) = \frac{9}{10}$.

Give necessary and sufficient conditions, in the form of inequalities, for the equation $\mathrm{f}(x) = c$ to have exactly n roots, where $n \geqslant 1$.

(See next page for some advice.)

Curve sketching, which involves visualising functions without the use of a graphical calculator, is considered to be a good test of mathematical understanding and crops up frequently in STEP.

You might wonder what the point is of doing the sketch without a calculator when it could be obtained much more easily with a calculator. The answer is that the final sketch itself is not particularly interesting or useful (except for the rest of the question) : who cares what it looks like? The interesting and useful part is the thought process that goes into working out how the sketch looks. If you use a calculator, the benefit of creating the sketch is lost to you for ever.

Often, as in the question above, the sketch has a purpose: it would be hard to answer the later parts of the question without having done the sketch. The sketch required, of $y = \dfrac{\lfloor x \rfloor}{x}$ *, is quite tricky, expecially for* $x < 0$ *: it will take some time to get it right. But having got it right, the rest of the question is not too hard. Overall, though, a pretty tough question.*

In the examination, there is a choice of questions (you only need to choose up to 6 questions), so those candidates who did not feel comfortable with the question simply did a different question.

The question below was **Q1 from STEP paper 1 in 2005**, and was explained in workshop 1.

Q2. 47231 is a five-digit number whose digits sum to $4 + 7 + 2 + 3 + 1 = 17$.

- **(i)** Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.
- **(ii)** How many five-digit numbers are there whose digits sum to 39?

The question below was **Q5 from STEP paper 1 in 2004**, and was explained in workshop 2.

Q3. The positive integers can be split into five distinct arithmetic progressions, as shown:

$$\begin{aligned} A:&\ \ 1,\ 6,\ 11,\ 16,\ \ldots \\ B:&\ \ 2,\ 7,\ 12,\ 17,\ \ldots \\ C:&\ \ 3,\ 8,\ 13,\ 18,\ \ldots \\ D:&\ \ 4,\ 9,\ 14,\ 19,\ \ldots \\ E:&\ \ 5, 10,\ 15,\ 20,\ \ldots \end{aligned}$$

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

(i) Prove that there are no positive integers x and y such that

$$x^2 + 5y = 243\,723\,.$$

(ii) Prove also that there are no positive integers x and y such that

$$x^4 + 2y^4 = 26\,081\,974\,.$$

The question below was **Q3 from STEP paper 1 in 2006**, and was explained in workshop 3.

Q4. In this question b, c, p and q are real numbers.

(i) By considering the graph $y = x^2 + bx + c$ show that $c < 0$ is a sufficient condition for the equation $x^2 + bx + c = 0$ to have distinct real roots. Determine whether $c < 0$ is a necessary condition for the equation to have distinct real roots.

(ii) Determine necessary and sufficient conditions for the equation $x^2 + bx + c = 0$ to have distinct positive real roots.

(iii) What can be deduced about the number and the nature of the roots of the equation $x^3 + px + q = 0$ if $p > 0$ and $q < 0$?

What can be deduced if $p < 0$ and $q < 0$? You should consider the different cases that arise according to the value of $4p^3 + 27q^2$.