**UNIVERSITY OF CAMBRIDGE**

**Faculty of Mathematics**

# MATHEMATICS FOR THE NATURAL SCIENCES WORKBOOK

This workbook is intended for students coming to Cambridge to study physical science options of the Natural Sciences Tripos, or the Computer Science Tripos.

July 14, 2016

# Introduction

Mathematics is an essential tool for all scientists. In the first year of the Natural Sciences Tripos, there are two mathematics courses: Mathematics (courses A or B) and Mathematical Biology. Most students taking physical science options follow Mathematics (courses A or B), Mathematical Biology is intended for students taking biological science options.

The choice whether to take the A or B course is made after discussion with your Director of Studies on arrival in Cambridge. For more information on the criteria which help in deciding which course to take and also on the content of the lecture courses, you can consult the course schedules for NST Mathematics at `http://www.maths.cam.ac.uk/undergradnst/`

The first section of this workbook contains core scientific mathematics questions based around material usually encountered in the core A-level syllabus - each question is labelled with the typical module in which it might be encountered. If you are planning to take Mathematics (courses A or B), which starts roughly at the level of the questions in this section, then we hope that you will work through these questions before arriving in Cambridge.

The second section of this workbook contains a few additional questions based around material usually encountered in the A-level "Further Pure" modules FP1 and FP2. This section is included mainly for the benefit of students who have not taken A-levels and are unsure whether they have covered the material assumed in the B course. If you find any of this material unfamiliar or difficult then you should probably take the A course.

You may have already mastered all the material, in which case this workbook will provide a useful set of revision problems. However, if any of the material in section I is new to you, or you get stuck on any of the questions in this section, then we suggest that you refer to an appropriate A–level textbook that covers core A–level material, especially if you find all of the questions on a particular topic problematic.

If you have difficulties with some questions, don't worry; when you get to Cambridge, tell your mathematics supervisor at the first opportunity and he or she will go through the relevant areas with you.

**At the end of the workbook there is a questionnaire. Please fill it in (there is no need to give your name) as it helps us to make sure that the lectures are pitched at the right level. The questionnaire will be collected during the first lecture.**

In addition to the workbook, a selection of problems especially chosen to help prepare for the study of Mathematics in the Cambridge Natural Science Tripos can be found on the NRICH website: `http://nrich.maths.org/6884`. These problems will greatly aid your mathematical thinking and are typically far more involved than those encountered at school. The NRICH website also includes an interactive workout designed to test fluency at A-level mathematics (`http://nrich.maths.org/7088`). You will also benefit from working through some of the rich scientific mathematics problems from stemNRICH: (`http://nrich.maths.org/advancedstem`). Here you can find questions on mathematical biology, physics, chemistry, engineering and the most important areas of applied mathematics. You will also find supporting articles to help you to understand the important role that mathematics will play in your study of science.

Any comments or queries about this workbook should be sent to *The Secretary of the Faculty Board of Mathematics, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA*, (e-mail: undergrad-office@maths.cam.ac.uk).

# SECTION I (CORE)

## ALGEBRA

Being fluent with the manipulation of algebra is the most essential aspect of mathematics in science.

### A1 Factorisation (C1)

Factorise into the product of two factors:

(i) $x^2 - 1$ ;    (ii) $a^2 - 4\,ab + 4\,b^2$ ;    (iii) $x^3 - 1$ .

### A2 Quadratic equations (C1)

Find the roots of:

(i) $x^2 - 5x + 6 = 0$    (ii) $x^2 + 2\,x = 0$

(iii) $x^2 - x - 1 = 0$    (iv) $x^4 - 3\,x^2 + 2 = 0$ .

### A3 Completing the square (C1)

By completing the square, find (for real $x$) the minimum values of:

(i) $x^2 - 2x + 6$ ;    (ii) $x^4 + 2x^2 + 2$ .

What is the minimum value of (i) in the domain $2 \leqslant x \leqslant 3$ ?

### A4 Inequalities (C1)

By factorizing a suitable polynomial, or otherwise, find the values of $x$ and $y$ which satisfy:

(i) $x^2 - 3x < 4$    (ii) $y^3 < 2y^2 + 3y$ .

### A5 Factor theorem (C2)

(i) Divide $x^3 + 5\,x^2 - 2\,x - 24$ by $(x + 4)$ and hence factorise it completely.

(ii) Use the factor theorem to factorise $t^3 - 7\,t + 6$ .

(iii) Simplify $\dfrac{x^3 + x^2 - 2\,x}{x^3 + 2\,x^2 - x - 2}$ .

**A6 Partial fractions (C4)**

Express the following in partial fractions:

(i) $\dfrac{2}{(x+1)(x-1)}$ (ii) $\dfrac{x+13}{(x+1)(x-2)(x+3)}$ (iii) $\dfrac{4x+1}{(x+1)^2(x-2)}$

(iv) $\dfrac{4x^2+x-2}{(x-1)(x^2+2)}$ .

## FUNCTIONS AND CURVE SKETCHING

You will need to be familiar with standard functions: polynomials, trigonometric functions, powers, exponentials and logarithms, along with combinations of these. You will need to be aware of the key features (zeros, asymptotes, limits, stationary points) of these functions and be able to sketch by hand combinations of these. You cannot rely on graphical calculators for this!

**FC1 Modulus function (C3)**

Sketch the curves given by:

(i) $y = |x|$ (ii) $y = 2 - |x|$

(iii) $y = |2 - |x||$ (iv) $y = (2 - |x|)(3 + |x|)$ .

**FC2 Transformations of functions (C3)**

Let $f(x) = x^2$. Sketch the following curves:

(i) $y = f(x)$ (ii) $y = 2f(x)$ (iii) $y = 2f(x) + 3$

(iv) $y = f(x-2)$ (v) $y = f(2x+1) + 3$ .

**FC3 Transformations of functions (C3)**

Repeat all parts of the previous question for the functions $f(x) = e^x$ and $f(x) = \ln x$.

**FC4 Trig and inverse trig functions (C3)**

Sketch the following curves, for suitable values of $x$:

(i) $y = \cos 2x$ (ii) $y = (\sin x)^2$ (iii) $y = 2\cot x$

(iv) $y = 2\cos^{-1} x$ (v) $y = e^{-x} \sin x$ .

**FC5 Composition of functions (C3)**

Let $f_1(x) = x^3$ and $f_2(x) = \tan x$. Sketch the following curves, taking particular care about the gradients of the functions when $y = 0$:
(Note that $f_1^{-1}(x)$ denotes the inverse function to $f_1(x)$.)

(i) $y = f_1(x)$ (ii) $y = f_2(x)$ (iii) $y = f_1^{-1}(x)$

(iv) $y = f_2^{-1}(x)$ (v) $y = f_1(f_2(x))$ (vi) $y = f_2(f_1(x))$.

What have you done to $f_2(x)$ to make $f_2^{-1}(x)$ well-defined in part (iv)?

**FC6 Curve sketching (C4)**

Sketch the curves in the $xy$ plane given by:

(i) $y = 2x^{2/3}$ (ii) $2x + 3y - 1 = 0$ (iii) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

(iv) $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ (v) $y = 3t + 4\,;\; x = t + 1$ (vi) $x = 2\cos t\,;\; y = 3\sin t$

(vii) $x = \tan t\,;\; y = \sec t$

In the last three parts, $t$ is a real parameter which ranges from $-\infty$ to $\infty$.

**GEOMETRY**

You will need to be familiar with the basic properties of lines, planes, triangles and circles.

**G1 Triangles (C2)**

(i) In triangle $ABC$, $AB = 1$, $BC = 1$ and $\angle A = \frac{\pi}{3}$ radians. Find $CA$ and $\angle B$.

(ii) In triangle $ABC$, $AB = 2$, $BC = 2$ and $AC = 3$. Find the angles of the triangle.

**G2 Circles (C2)**

Find, for a sector of angle $\frac{\pi}{3}$ radians of a disc of radius 3:

(i) the length of the perimeter; and (ii) the area.

**G3 Lines in 3D (C4)**

Find the angle between the lines $x = y = z$ and $x = y = 2z + 1$ and determine whether the lines intersect.

**SEQUENCES AND SERIES**

There are three different sorts of series you will commonly encounter in science: Firstly, you will need to use the series expansions for common functions; secondly you will need to be able expand brackets using the binomial theorem; thirdly you will need to be able to sum arithmetic or geometric progressions

**SS1 Arithmetic progressions (C1)**

An arithmetic progression has third term $\alpha$ and ninth term $\beta$. Find the sum to thirty terms.

**SS2 Powers (C1)**

Simplify

$$\frac{x^{-\frac{1}{5}} \times \left(x^{\frac{2}{3}}\right)^6}{x \times \sqrt[2]{x^5} \times \sqrt[5]{x^2}} \, .$$

**SS3 Binomial expansions (C2)**

Expand the following expressions, using the binomial expansion, as far as the fourth term:

$$\text{(i)} \quad (1+x)^3 \qquad\qquad \text{(ii)} \quad (2+x)^4 \qquad\qquad \text{(iii)} \quad \left(2+\frac{3}{x}\right)^5 .$$

**SS4 Logarithms (C2)**

(i) If $3 = 9^{-x}$ find $x$.

(ii) If $\log_a b = c$, show that $c = \frac{\log_\alpha b}{\log_\alpha a}$ for any base $\alpha$.

(iii) Find $x$ if $16 \log_x 3 = \log_3 x$.

**SS5 Arithmetic and geometric progressions (C2)**

Prove that $\sum\limits_1^N n \; = \; \frac{1}{2} N(N+1)$.

Evaluate:

(i) the sum of the odd integers from 11 to 99 inclusive

(ii) $\sum\limits_{n=1}^{5} (3n+2)$

(iii) $\sum\limits_{n=0}^{N} (an+b)$ ($a$ and $b$ are constants)

(iv) $\sum\limits_{r=0}^{10} 2^r$

(v) $\sum\limits_{n=0}^{N} ar^{2n}$ ($a$ and $r$ are constants).

**SS6 Iterative sequences (C2)**

The sequence $u_n$ satisfies $u_{n+1} = ku_n$, where $k$ is a fixed number, and $u_0 = 1$. Express $u_n$ in terms of $k$. Describe the behaviour of $u_n$ for large $n$ in the different cases that arise according to the value of $k$.

**SS7 Binomial expansion for rational powers (C4)**

Find the first four terms in the expansions in ascending powers of $x$ of the following expressions, stating for what values of $x$ the expansion is valid in each case:

(i) $(1+x)^{\frac{1}{2}}$ (ii) $(2+x)^{\frac{2}{5}}$ (iii) $\dfrac{(1+2x)^{\frac{1}{2}}}{(2+x)^{\frac{1}{3}}}$ .

**SS8 Composition of approximations (?C4)**

Given that, for small $\theta$, $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, find an approximation, ignoring powers of $\theta$ greater than 3, for $\sin(\frac{1}{2}\theta)\cos\theta + \sec 2\theta$ .

**TRIGONOMETRY**

Trigonometrical functions are of fundamental importance in science. You will need to know special values at which sin(x), cos(x) and tan(x) are zero, have turning points or tend to infinity. There are many trigonometric identities and many formulae for double angles. You will need to be aware of these and either know them or be able to work them out as required.

**T1 Solving trig equations (C2)**

Find the four values of $\theta$ in the range 0 to $2\pi$ that satisfy the equation $2\sin^2\theta = 1$ .

**T2 Trig identities (C3)**

Prove that $\dfrac{\cot^2 x + \sin^2 x}{\cos x + \operatorname{cosec} x} = \operatorname{cosec} x - \cos x$ .

**T3 Trig identities (C3)**

By writing $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ , use trigonometric identities to evaluate:

(i) $\cos\dfrac{\pi}{12}$ ; (ii) $\sin\dfrac{\pi}{12}$ ; (iii) $\cot\dfrac{\pi}{12}$ .

**T4 Trig identities (C3)**

If $t = \tan\frac{1}{2}\theta$ , express the following in terms of $t$: (i) $\cos\theta$ ; (ii) $\sin\theta$ ; (iii) $\tan\theta$ .

**T5 Trig identities (C3)**

Simplify $\tan(\arctan\frac{1}{3} + \arctan\frac{1}{4})$ .

**T6 Trig identities (C3)**

If $A$, $B$ and $C$ are the angles of a triangle, prove that

$$\cos\left(\frac{B-C}{2}\right) - \sin\left(\frac{A}{2}\right) = 2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) .$$

**T7 Solving trig equations (C3)**

Write $\sqrt{3}\sin\theta + \cos\theta$ in the form $A\sin(\theta+\alpha)$, where $A$ and $\alpha$ are to be determined.

**T8 Solving trig equations (C3)**

Find the values of $\theta$ in the range 0 to $2\pi$ which satisfy the equation

$$\cos\theta + \cos 3\theta = \sin\theta + \sin 3\theta\ .$$

## VECTORS

Vectors are of fundamental importance in all branches of mathematics and it is good to become comfortable with manipulating them. These questions involve the basic ideas of lines and scalar products in 3D, although the use of vectors goes far beyond this.

**V1 Scalar products in 3D (C4)**

Consider the four vectors

$$\mathbf{A} = \begin{pmatrix} 16 \\ -6 \\ 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 \\ 14 \\ -9 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -15 \\ 7 \\ 4 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 12 \\ 12 \\ 1 \end{pmatrix}.$$

(i) Order the vectors by magnitude.

(ii) Use the scalar product to find the angles between the pairs of vectors (a) $\mathbf{A}$ and $\mathbf{B}$, (b) $\mathbf{B}$ and $\mathbf{C}$.

**V2 Vector equation of lines (C4)**

Show that the points with position vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix},$$

lie on a straight line and give the equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

## DIFFERENTIATION

Differentiation measures the rate of change of a quantity; as such differentiation is very important in science. You will need to know how to differentiate standard functions, products, quotients and functions of a function (using the chain rule).

**D1 Stationary points (C1)**

Find the stationary points of the following functions, stating whether they are local maxima, minima or points of inflexion:

(i) $y = x^2 + 2$ (ii) $y = x^3 - 3x + 3$

(iii) $y = x^3 - 3x^2 + 3x$ (iv) $y = x^3 + 3x + 3$.

Sketch the graphs of the functions.

**D2 Differentiation from first principles (C1)**

Calculate the derivative of $y = x^2 + 1$ from first principles (i.e. by considering the derivative of a function as the limit of the gradient of a chord).

**D3 Chain rule and product rule (C3)**

Using the chain and product rules etc., find the derivatives of:

(i) $y = \sin(x^2)$ (ii) $y = a^x$(hint: take logs) (iii) $y = \ln(x^a + x^{-a})$

(iv) $y = x^x$ (v) $y = \sin^{-1} x$ .

where $a$ is a positive constant.

**D4 Implicit differentiation (C4)**

If $y + e^y = x + x^3 + 1$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $y$ and $x$.

**D5 Implicit differentiation (C4)**

If $y = \dfrac{t+1}{t-2}$, and $x = \dfrac{2t+1}{t-3}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $t = 1$.

**INTEGRATION**

Integration is used to find areas under curves and more generally as a summation tool. You will need to know the integral of standard functions and be able to integrate function by parts and by substitution.

**I1 Integration techniques (C4)**

Find the following indefinite integrals (stating the values of $x$ for which the integrand is a real function):

(i) $\displaystyle\int \frac{1}{2+x^2}\ dx$ (set $x = \sqrt{2}\tan\theta$) (ii) $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$ (set $x - 1 = 2\sin\theta$)

(iii) $\displaystyle\int \frac{1}{x\sqrt{1-x}}\ dx$ (iv) $\displaystyle\int \ln x\ dx$ .

**I2 Integration techniques (C4)**

Evaluate the following definite integrals:

(i) $\displaystyle\int_0^L xe^{-x}\ dx$ (ii) $\displaystyle\int_0^{\pi/2} \sin 3\theta\ \cos\theta\ d\theta$

(iii) $\displaystyle\int_0^1 \frac{x^2+1}{x^3+3x+2}\,\mathrm{d}x$ (iv) $\displaystyle\int_0^{\pi/2} \frac{1}{3+5\cos\theta}\ d\theta$ [ use $t = \tan(\frac{1}{2}\theta)$ ] .

In part (i), can you suggest what happens as $L \to \infty$?

**DIFFERENTIAL EQUATIONS**

Equations of science often involve the rate of change of a quantity; solving equations involving differentials is important. You will need to be able to solve linear second order differential equations with constant coefficients and simple first order differential equations.

**DE1 Separable first order ODEs (C4)**

Solve the following differential equation:

$$x\,\frac{dy}{dx} + (1-y^2)\ =\ 0\,; \qquad\qquad y = 0 \ \text{ when } \ x = 1.$$

# SECTION 2 (FURTHER PURE)

## COMPLEX NUMBERS

### C1 Basic manipulations

(i) determine the real and imaginary parts of

$$\frac{1+i}{2-i}$$

(ii) Find the roots of the quadratic equation $z^2 - 2z + 2 = 0$. Determine the modulus and argument of each root. Plot the roots on an Argand diagram.

### C2 Further properties

(i) Use de Moivre's theorem to express $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.

(ii) Sketch the loci $|z-i| = 2$ and $|z+i| = |z-2|$.

## MATRICES

### M1 basic properties

Calculate $\mathbf{A}+\mathbf{B}$, $\mathbf{AB}$ and $\mathbf{BA}$ for

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$$

### M2 non-commutativity

Find matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} = 0$ and $\mathbf{BA} \neq 0$.

### M3 transformations

A linear transformation is described by the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Show that this transformation is the composition of a rotation and a scaling.

## SERIES

### SE1 Summation of series

Sum the following series

$$\sum_{r=1}^{n} r^2 \qquad \sum_{r=1}^{n} r(r^2+2)$$

### SE2 Method of differences

Use partial fractions to sum the series

$$\sum_{r=1}^{n} \frac{1}{r(r+1)}$$

## MATHEMATICAL INDUCTION

### IN1 Sequences

If $a_{n+1} = 3a_n + 4$ and $a_1 = 1$ then deduce a formula for $a_n$ for any $n \geqslant 1$. Use mathematical induction to prove your result.

### IN2 Integration

Use mathematical induction to prove that, for a non-negative integer $n$,

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

## HYPERBOLIC FUNCTIONS

### H1 Basic properties

State the definitions of $\sinh x$ and $\cosh x$. Prove that

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh 2x = 2\sinh x \cosh x$$

### H2 Differentiation

Prove that

$$\frac{d}{dx}\tanh x = \mathrm{sech}^2 x$$

# ANSWERS TO SECTION 1

**A1**

(i) $(x+1)(x-1)$; (ii) $(a-2b)^2$; (iii) $(x-1)(x^2+x+1)$.

**A2**

(i) 3, 2 ; (ii) 0, −2 ; (iii) $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ ; (iv) $\pm 1,\ \pm\sqrt{2}$.

**A3**

(i) 5 ; (ii) 2 . The minimum is 6 .

**A4**

(i) $-1 < x < 4$; (ii) $y < -1$ and $0 < y < 3$.

**A5**

(i) $(x^3+5x^2-2x-24) \div (x+4) = x^2+x-6$ and hence $(x^3+5x^2-2x-24) = (x+4)(x+3)(x-2)$;

(ii) $(t-1)(t-2)(t+3)$;

(iii) $\frac{x}{x+1}$

**A6**

(i) $\dfrac{1}{(x-1)} - \dfrac{1}{(x+1)}$, (ii) $\dfrac{1}{x-2} - \dfrac{2}{(x+1)} + \dfrac{1}{(x+3)}$, (iii) $\dfrac{1}{(x+1)^2} - \dfrac{1}{x+1} + \dfrac{1}{(x-2)}$,

(iv) $\dfrac{1}{(x-1)} + \dfrac{(3x+4)}{(x^2+2)}$.

**FC5**

One has to restrict the range of $x$ (to $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, say) to make the function $f_2(x)$ one-to-one (so that it has a unique value).

**G1**

(i) $AC = 1$, $B = C = \dfrac{\pi}{3}$ radians; (ii) $\cos C = \cos A = \dfrac{3}{4}$, $\cos B = -\dfrac{1}{8}$.

**G2**

(i) $6 + \pi$; (ii) $\dfrac{3\pi}{2}$.

**G3**

$\arccos\left(\dfrac{5}{3\sqrt{3}}\right)$; they intersect at $x = y = z = -1$.

**SS1**

$\frac{1}{2}(125\beta - 65\alpha)$.

**SS2**

$x^{-\frac{1}{10}}$.

**SS3**

(i) $1 + 3x + 3x^2 + x^3$, (ii) $16 + 32x + 24x^2 + 8x^3$

(iii) $32 + \dfrac{240}{x} + \dfrac{720}{x^2} + \dfrac{1080}{x^3}$

**SS4**

(i) $x = -\frac{1}{2}$; (iii) $x = 81$ or $x = 1/81$.

**SS5**

Proof: (e.g.) take average and multiply by number of terms.

(i) 2475; (ii) 55; (iii) $\dfrac{a}{2}N(N+1) + b(N+1)$; (iv) $2^{11} - 1$; (v) $a\left(1 - r^{2N+2}\right)\left(1 - r^2\right)^{-1}$.

**SS6**

$u_n = k^n$. If $|k| < 1$, $u_n \to 0$; if $k = 1$, $u_n = 1$; if $k = -1$, $u_n$ oscillates; if $k > 1$, $u_n \to \infty$; if $k < -1$, $u_n$ oscillates, with $|u_n| \to \infty$.

**SS7**

(i) $1 + \frac{1}{2}x - \dfrac{1}{8}x^2 + \dfrac{1}{16}x^3, \quad |x| < 1$

(ii) $2^{\frac{2}{5}}\left(1 + \dfrac{x}{5} - \dfrac{3x^2}{100} + \dfrac{x^3}{125}\right), \quad |x| < 2;$

(iii) $\dfrac{1}{\sqrt[3]{2}}\left(1 + \dfrac{5x}{6} - \dfrac{11x^2}{18} + \dfrac{50x^3}{81}\right), \quad |x| < \frac{1}{2}.$

**SS8**

$1 + \frac{1}{2}\theta + 2\theta^2 - \dfrac{13}{48}\theta^3.$

**T1**

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$

**T3**

(i) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$; (ii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$; (iii) $2+\sqrt{3}$.

**T4**

(i) $\dfrac{1-t^2}{1+t^2}$; (ii) $\dfrac{2t}{1+t^2}$; (iii) $\dfrac{2t}{1-t^2}$.

**T5**

$\frac{7}{11}.$

**T7**

$2\sin\left(\theta+\frac{\pi}{6}\right).$

**T8**

$\frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{3\pi}{2}, \frac{13\pi}{8}.$

**V1**

(i) $|\mathbf{A}| = |\mathbf{B}| > |\mathbf{C}| > |\mathbf{D}|$;

(ii) (a) $\arccos\left(\dfrac{-29}{293}\right)$, (b) $\arccos\left(\dfrac{2}{\sqrt{293}\sqrt{290}}\right)$;

**V2**

(i) $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$

**D1**

(i) $(x, y) = (0, 2)$, a local minimum;
(ii) $(x, y) = (1, 1)$, a local minimum; $(x, y) = (-1, 5)$, a local maximum;
(iii) $(x, y) = (1, 1)$, a point of inflexion;
(iv) no stationary points.

**D3**

(i) $2x\cos(x^2)$, (ii) $a^x \log_e a$, (iii) $\dfrac{a(x^{a-1} - x^{-a-1})}{(x^a + x^{-a})}$,

(iv) $x^x(\ln x + 1)$, (v) $\dfrac{1}{\sqrt{1-x^2}}$.

**D4**

$\dfrac{1+3x^2}{1+e^y}$.

**D5**

$\dfrac{12}{7}$.

**I1**

(i) $\dfrac{1}{\sqrt{2}} \arctan\left(\dfrac{x}{\sqrt{2}}\right) +$ constant.

(ii) $\arcsin\left(\dfrac{x-1}{2}\right) +$ constant.

[Hint: write $3 + 2x - x^2$ as $4 - (x-1)^2$ and then substitute $x = 1 + 2\sin\theta$ ]

(iii) $\log_e\left(\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right) +$ constant. [Hint: substitute $y = \sqrt{1-x}$ ]

(iv) $x\log_e x - x +$ constant.

**I2**

(i) $1 - (1+L)e^{-L}$, (ii) $\frac{1}{2}$, (iii) $\dfrac{1}{3}\log_e 3$, (iv) $\dfrac{1}{4}\log_e 3$.

**DE1**

$$y = \frac{1-x^2}{1+x^2}.$$

## QUESTIONNAIRE

*Please tick one box in each table for each question*

| Table A | | |
|---|---|---|
| Material covered at school | | |
| | Yes | No |
| A1 | | |
| A2 | | |
| A3 | | |
| A4 | | |
| A5 | | |
| A6 | | |
| FC1 | | |
| FC2 | | |
| FC3 | | |
| FC4 | | |
| FC5 | | |
| FC6 | | |
| G1 | | |
| G2 | | |
| G3 | | |
| SS1 | | |
| SS2 | | |
| SS3 | | |
| SS4 | | |
| SS5 | | |
| SS6 | | |
| SS7 | | |
| SS8 | | |

| Table B | | | | | |
|---|---|---|---|---|---|
| Difficulty of questions | | | | | |
| | Easy | | | | Difficult |
| A1 | | | | | |
| A2 | | | | | |
| A3 | | | | | |
| A4 | | | | | |
| A5 | | | | | |
| A6 | | | | | |
| FC1 | | | | | |
| FC2 | | | | | |
| FC3 | | | | | |
| FC4 | | | | | |
| FC5 | | | | | |
| FC6 | | | | | |
| G1 | | | | | |
| G2 | | | | | |
| G3 | | | | | |
| SS1 | | | | | |
| SS2 | | | | | |
| SS3 | | | | | |
| SS4 | | | | | |
| SS5 | | | | | |
| SS6 | | | | | |
| SS7 | | | | | |
| SS8 | | | | | |

## QUESTIONNAIRE

*Please tick one box in each table for each question*

| **Table A** | | |
|---|---|---|
| Material covered at school | | |
| | Yes | No |
| T1 | | |
| T2 | | |
| T3 | | |
| T4 | | |
| T5 | | |
| T6 | | |
| T7 | | |
| T8 | | |
| V1 | | |
| V2 | | |
| D1 | | |
| D2 | | |
| D3 | | |
| D4 | | |
| D5 | | |
| I1 | | |
| I2 | | |
| DE1 | | |

| **Table B** | | | | | |
|---|---|---|---|---|---|
| Difficulty of questions | | | | | |
| | Easy | | | | Difficult |
| T1 | | | | | |
| T2 | | | | | |
| T3 | | | | | |
| T4 | | | | | |
| T5 | | | | | |
| T6 | | | | | |
| T7 | | | | | |
| T8 | | | | | |
| V1 | | | | | |
| V2 | | | | | |
| D1 | | | | | |
| D2 | | | | | |
| D3 | | | | | |
| D4 | | | | | |
| D5 | | | | | |
| I1 | | | | | |
| I2 | | | | | |
| DE1 | | | | | |

Please indicate how you made use of NRICH materials during your application and preparation.

| Webpage | How useful/interesting did you find this resource? | | | |
|---|---|---|---|---|
| | Didn't use at all | Not particularly | Quite | Very |
| Mathematical Preparation for the Cambridge Natural Sciences Tripos `http://nrich.maths.org/6884` | | | | |
| Advanced stemNRICH `http://nrich.maths.org/advancedstem` | | | | |
| Interactive workout `http://nrich.maths.org/7088` | | | | |