Tensors

T. J. Crawford, J. Goedecke, P. Haas, E. Lauga, J. Munro, J. M. F. Tsang

July 14, 2016

1 Relevant courses

The relevant Cambridge undergraduate course is IA Vector Calculus.

2 Books


3 Notes

3.1 Definition and examples

Consider orthogonal right-handed bases \{e_i\} and \{e'_i\} in \mathbb{R}^3 with corresponding Cartesian coordinates \{x_i\} and \{x'_i\}. Then a vector \(x \in \mathbb{R}^3\) can be written as

\[ x = x_i e_i = x'_i e'_i \]

(using summation convention throughout these notes).

These two bases are related by a rotation:

\[ e'_i = R_{ip} e_p \quad \text{and} \quad x'_i = R_{ip} x_p \]

where \(R\) is a rotation matrix, so

• \(R\) is orthogonal: \(R_{ip} R_{jp} = R_{qp} R_{ij} = \delta_{ij}\), and
• \(\det R = 1\).

Tensors are geometrical objects which obey a generalised form of this transformation rule. By definition, a tensor \(T\) of rank \(n\) has components \(T_{i_1 \ldots i_n}\) (with \(n\) indices) with respect to each basis \(\{e_i\}\) or coordinate system \(\{x_i\}\), obeying the tensor transformation rule

\[ T'_{i_1 \ldots k} = R_{ip} R_{jq} \ldots R_{kr} T_{pq \ldots r}. \]

under a change of basis.

Examples  Here are some examples of the tensor transformation rule for different ranks:

• Rank 0: \(T' = T\). Rank 0 tensors are scalars.
• Rank 1: \(T'_{i} = R_{ip} T_{p}\). As we have seen, these are vectors.
• Rank 2: \(T'_{ij} = R_{ip} R_{jq} T_{pq}\). These are matrices which represent linear maps or quadratic forms.
Special cases  The tensors $\delta_{ij}$ and $\epsilon_{ijk}$ are tensors of rank 2 and 3 respectively, with the special property that their components are unchanged under any change of basis:

$$R_{ip}R_{jq}\delta_{pq} = R_{iq}R_{jp} = \delta_{ij}$$

and

$$R_{ip}R_{jq}R_{kr}\epsilon_{pqr} = (\det R)\epsilon_{ijk} = \epsilon_{ijk}.$$

Symmetric and antisymmetric tensors  A tensor of rank $n$ obeying $T_{ijp...q} = \pm T_{jip...q}$ is said to be symmetric/antisymmetric in the indices $i$ and $j$. A tensor is said to be totally symmetric/antisymmetric if it is symmetric/antisymmetric under any such swap of indices.

So, $\delta_{ij}$ is totally symmetric and $\epsilon_{ijk}$ is totally antisymmetric.

In $\mathbb{R}^3$, any totally antisymmetric tensor of rank 3 takes the form $T_{ijk} = \lambda\epsilon_{ijk}$ for some $\lambda$. There is no totally antisymmetric tensor of rank $n > 3$ (unless all components are zeros).

4  Exercises

4.1  Basic properties

Show each of the following:

- If $T$ and $S$ are both tensors of rank $n$, then $(T + S)_{ij...k} = T_{ij...k} + S_{ij...k}$. (Hint: Use the transformation rule.)
- If $\alpha$ is a scalar, then $(\alpha T)_{ij...k} = \alpha T_{ij...k}$. 
- Hence, any linear combination of rank $n$ tensors is itself a rank $n$ tensor.
- If $T$ and $S$ are tensors of rank $n$ and $m$ respectively, then the tensor product

$$\left(T \otimes S\right)_{ij...kpq...r} = T_{ij...k}S_{pq...r}$$

is a tensor of rank $n + m$.
- If $u_1, v, \ldots, w$ are $n$ vectors, then $T_{ij...k} = u_i v_j \ldots w_k$ defines a tensor of rank $n$.
- If $T_{ijp...q}$ is a tensor of rank $n$, then $S_{p...q} = \delta_{ij}T_{ijp...q}$ is a tensor of rank $n - 2$. (Note that contracting on a different pair of indices in general results in a different tensor.)
- The trace of a matrix, $T_{ii}$, is a rank 0 tensor.

4.2  Further exercises

Let $u_i(x)$ be a vector field and let $\sigma_{ij}(x)$ be a second-rank tensor field. Show that:

- $\partial u_i / \partial x_j$ transforms as a rank 2 tensor.
- $\nabla \cdot u = \partial u_i / \partial x_i$ is a scalar.
- $\partial \sigma_{ij} / \partial x_j$ transforms as a vector.

Let $T$ be a tensor of rank 3, satisfying

$$T_{ijk} = T_{jik} \text{ and } T_{ijk} = -T_{ikj}.$$

Show that $T_{ijk} = 0$.

Let $T$ be a tensor of rank 4, satisfying

$$T_{ijkl} = -T_{jikl} = -T_{ijlk} \text{ and } T_{ijij} = 0.$$

Show that

$$T_{ijkl} = \epsilon_{ijp}\epsilon_{klq}S_{pq} - T_{rqrp}.$$