1 Relevant courses

The relevant Cambridge undergraduate courses are IB Fluid Dynamics, II Fluid Dynamics and II Mathematical Biology.

2 Books

The principles of scaling analysis, and a number of examples, are given in *Elementary Fluid Dynamics* by D. J. Acheson, OUP 1990.

3 Notes

3.1 The heat equation

Consider the heat equation

\[ \frac{\partial u}{\partial t} = \alpha \nabla^2 u \]

where \( \alpha \) is the thermal diffusivity (related to the thermal conductivity of a material \( k \) by \( \alpha = k/(c_p \rho) \)).

Specifically, consider this equation on the one-dimensional domain \( x \in [0, L] \) for \( t > 0 \) (so that \( \nabla^2 = \frac{d^2}{dx^2} \)). The initial and boundary conditions are

\[ u(x, 0) = f(x) \text{ for some given } f \]

and

\[ u(0, t) = u(L, t) \text{ for all } t. \]

What is the timescale for heat to decay?

There are two approaches for analysing this problem.

**Approach 1: Exact solution** We solve the equation exactly using separation of variables. Looking for \( u(x, t) \) as a superposition of solutions of the form \( T(t)X(x) \), we find that the solution takes the form

\[ u(x, t) = \sum_{n=1}^{\infty} D_n \sin \left( \frac{n\pi x}{L} \right) \exp \left( -\frac{n^2 \pi^2 \alpha t}{L^2} \right). \]

Therefore, the decay happens over a timescale \( T \propto L^2/\alpha \).
Approach 2: Dimensional/scaling analysis  Let us write \( u(x, t) = U \hat{u}(x, t) \), where \( U \) is a ‘typical’ temperature. Also, write \( t = T \hat{t} \) and \( x = L \hat{x} \). We are interested in finding the timescale \( T \). The hatted quantities are all nondimensional.

Substituting these into the heat equation, and dropping hats, one gets
\[
\frac{U}{T} \frac{U_t}{U} = \frac{\alpha U}{L^2} \frac{U_{xx}}{U}.
\]
Since the dimensions must match up,
\[
\frac{U}{T} \propto \frac{\alpha U}{L^2}
\]
and so \( T \propto L^2/\alpha \) as before.

3.2 The Reynolds number

Consider the Navier-Stokes equations
\[
\rho \frac{Du}{Dt} = -\nabla p + \mu \nabla^2 u \quad \text{and} \quad \nabla \cdot u = 0
\]
supposing that there are no body forces.

Let \( u = U \hat{u}, x = L \hat{x}, \ t = T \hat{t} \) and \( p = P \hat{p} \), where \( T \) is the advective timescale, \( T = L/U \) (so that the \( \frac{\partial u}{\partial t} \) and \( u \cdot \nabla u \) terms are scaled equally). Putting these into the Navier-Stokes equations, and dropping hats, we get
\[
\frac{\rho U^2 Du}{L} \frac{Du}{Dt} = -\frac{P}{L} \nabla P + \frac{\mu U}{L^2} \nabla^2 u.
\]
The Reynolds number is defined as the ratio of the inertia term to the viscous term:
\[
\text{Re} = \frac{\text{inertia}}{\text{viscosity}} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{\rho L U}{\mu} = \frac{LU}{\nu}.
\]
We can proceed in two different ways:

- Dividing by the viscous scale gives us
  \[
  \text{Re} \frac{Du}{Dt} = -\frac{P}{\rho U^2} \nabla P + \nabla^2 u.
  \]
  For small \( \text{Re} \), we choose the viscous pressure scale \( P \sim \frac{\mu U}{L} \), and we get the Stokes equations
  \[
  \nabla p = \nabla^2 u
  \]
  which hold at low Reynolds numbers (i.e. for viscous flows).

- Alternatively, dividing by the inertia scale gives us
  \[
  \frac{Du}{Dt} = -\frac{P}{\rho U^2} \nabla P + \frac{1}{\text{Re}} \nabla^2 u.
  \]
  For large \( \text{Re} \), we choose the inviscid pressure scale \( P \sim \rho U^2 \), to get the inviscid Euler equations
  \[
  \frac{Du}{Dt} = -\nabla P.
  \]

4 Exercises

4.1 Exercise 1

A simple model of two competing species eating the same food takes the form
\[
\frac{dN_1}{dt} = r_1 N_1 \left( 1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_2} \right),
\]
\[
\frac{dN_2}{dt} = r_2 N_2 \left( 1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_1} \right),
\]
where \( N_1 \) and \( N_2 \) are the population sizes. Rescale the equations to simplify them, and show that the solutions depend only on \( \rho = r_2/r_1, b_{12} \) and \( b_{21} \).
### 4.2 Exercise 2

The concentration of a chemical \( C(x,t) \) satisfies the nonlinear diffusion equation

\[
\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left( D(C) \frac{\partial C}{\partial x} \right)
\]

and the condition \( \int_{-\infty}^{\infty} C(x,t) \, dx = M \). The diffusivity is given by \( D(C) = kC^p \), and \( M, k \) and \( p \) are positive constants.

Use dimensional analysis to find a suitable space-like scale \( \xi \) and a space-independent \( \eta \) for the similarity solution of the form

\[
C(x,t) = \eta F(\xi)
\]

Use this form to seek the solution initially localised to the origin, and show that \( F \) is of the form

\[
F(\xi) = \begin{cases} 
(A - \frac{p}{2x^p}\xi^2)^{1/p} & \text{for } |\xi| < \xi_0 \\
0 & \text{otherwise}
\end{cases}
\]

for some \( A \) and \( \xi_0 \). For the case when \( p = 2 \), find \( A \) and \( \xi_0 \).

### 4.3 Exercise 3: Flow in a 2D thin layer

Consider a flow in a 2D domain where \( x \sim L \) and \( y \sim \delta L \), where \( \delta \ll 1 \) so the domain is thin. How do \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) scale?

If \( u \sim U \), explain why \( v \sim \delta U \), and explain why the advective timescale \( T \) is proportional to \( L/U \).

Rescale the Navier-Stokes equations, taking an advective timescale and choosing the pressure scale \( P \) so that it always balances the \( x \)-momentum equation. Show that three regimes are possible, depending on how large \( Re \) is compared to \( \delta \):

- If \( \delta^2 Re \ll 1 \) then \( P \sim \frac{\mu U}{\delta L} \), and
  \[
  0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad 0 = -\frac{\partial p}{\partial y}.
  \]
  This is the lubrication regime.

- If \( \delta^2 Re \sim 1 \) then again \( P \sim \frac{\mu U}{\delta L} \), but now
  \[
  \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad 0 = -\frac{\partial p}{\partial y}.
  \]
  These are the unsteady boundary layer equations. They represent the flow of a low-viscosity fluid in a thin layer near a no-slip boundary; the thickness of the boundary layer is controlled by the viscosity of the fluid.

- For \( \delta^2 Re \gg 1 \), one has \( P \sim \rho U^2 \), and
  \[
  \frac{Du}{Dt} = -\frac{\partial p}{\partial x} \quad \text{and} \quad 0 = -\frac{\partial p}{\partial y}.
  \]
  This is the shallow water regime. This regime can be used to model the flow of low-viscosity fluids in chutes, rivers or even oceans (provided that horizontal lengths are far greater than the depth).

### 4.4 Exercise 4: Decay of vorticity

Writing \( \omega = \nabla \times u \) for the vorticity, and using the identities

\[
(u \cdot \nabla)u = (\nabla \times u) \times u + \nabla \left( \frac{1}{2} |u|^2 \right)
\]

and

\[
\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u),
\]

3
show that

\[
\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nu \nabla^2 \omega
\]

provided that body forces are conservative. This is the \textit{vorticity equation}.

Why does the \((\omega \cdot \nabla)u\) term vanish in the 2D case?

Show that vorticity decays over a lengthscale \( L \propto \frac{\nu}{U} \).