Mathematical Tripos
Part III Guide to Courses 2017-2018

The Faulkes Institute of Geometry, completed in January 2002
Mathematical Tripos
Part III Lecture Courses in 2017-2018

Department of Pure Mathematics
& Mathematical Statistics

Department of Applied Mathematics
& Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is no requirement that students study only courses offered by one Department.

- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 [credit units], while a 24 lecture course is equivalent to 3 [credit units]. Please note that certain courses are non-examinable, and are indicated as such after the title. Some of these courses may be the basis for Part III essays.

- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.

- The courses described in this document apply only for the academic year 2017-18. Details for subsequent years are often broadly similar, but not necessarily identical. The courses evolve from year to year.

- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do not constitute definitive syllabuses. The lectures and associated course materials as offered in this academic year define the syllabus. Each course lecturer has discretion to vary the material covered.

- Some courses have no writeup available at this time, in which case you will see "No description available" in place of a description. Course descriptions will be added to the online version of the Guide to Courses as soon as they are provided by the lecturer. Until then, the descriptions for the previous year (available at http://www.maths.cam.ac.uk/postgrad/mathiii/courseguide.html) may be helpful in giving a rough idea of course content, but beware of the comments in the preceding item on what defines the syllabus.
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Commutative Algebra (M24)
A. Thompson

The aim of the course is to give an introduction to the theory of commutative Noetherian rings and modules, a theory that is an essential ingredient in algebraic geometry, algebraic number theory and representation theory.

I plan to cover the theory of ideals for Noetherian and Artinian rings, localisations and completions, integral closure, valuation rings and Dedekind rings, and dimension theory. Time permitting, I also hope to cover some or all of: projective and injective modules, resolutions, the Koszul complex, and (co)homology.

Pre-requisites

It will be assumed that you have attended a first course on ring theory, e.g., IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields will be helpful but not necessary.

Literature


The basic text that we will use for much of the course is Atiyah and Macdonald, but it doesn’t go into much detail and many results are left to the exercises. Sharp fills in some of the gaps, but neither book goes far enough. Both Kaplanksy and Matsumura cover the additional material, though Matsumura is a bit tough as an introduction. Reid’s book is a companion to one on algebraic geometry, which influences his choice of topics and examples; I may dip into it from time to time to add a bit of colour. Bourbaki is encyclopaedic.

Additional support

Four examples sheets will be provided, with supporting examples classes.

Lie Algebras and their representations (M24)
Beth Romano

This course is an introduction to the properties and representations of semisimple complex Lie algebras. The structure and representation theory of semisimple Lie algebras is one of the most beautiful and wide-reaching subjects in mathematics. It has applications to number theory, topology, algebraic geometry,
and theoretical physics, to name a few examples, as well as to the representation theory of real and \( p \)-adic groups.

Lie algebras arise as tangent spaces to certain differential manifolds called Lie groups, yet they can be defined purely algebraically. The representation theory of a complex simple Lie algebra, which can be understood in terms of combinatorial data, completely determines that of a corresponding group. Understanding this data (e.g. roots, weights, and Weyl groups) will be at the heart of this course.

The following is an outline of the topics that will be covered in the course:

1. The basic structure theory of semisimple Lie algebras.
4. (If time permits) Crystals and Littelmann paths.
5. (If time permits) Chevalley bases and a brief introduction to Chevalley groups.

Desirable Previous Knowledge

The only real prerequisite is linear algebra, though it will be beneficial to have some familiarity with tensor products, symmetric powers, and exterior powers of vector spaces. The Part II representation theory course (or its equivalent) will be useful but is not required.

Reading to complement course material


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Representation Theory (L24)

E. Giannelli

Overview

The representation theory of the symmetric group \( S_n \) is a classical subject that, from the foundational work of Frobenius, Schur and Young, has developed into a richly diverse area, with important connections across algebra, computer science, statistical mechanics and theoretical physics. In modern representation theory of finite groups, symmetric groups provide evidence for some of the deepest open problems, such as the McKay Conjecture (see [5]).
Course description

This course aims to give graduate students an introduction to the ordinary and the modular representation theory of symmetric groups.

To do this we choose an objective: the McKay Conjecture for symmetric groups.

Starting from basic ideas and definitions, we will cover all the necessary algebraic and combinatorial techniques needed to describe the approach to the above mentioned key problem. This will also give the interested student a sufficient background to tackle other open problems in this rich area of mathematics.

Here is a brief summary of the main topics we will cover in the course.

- Combinatorics of partitions I: Young diagrams and tableaux.
- Specht and simple modules for symmetric groups.
- Standard Basis Theorem for Specht modules.
- Hook-length formula for dimensions.
- Combinatorics of partitions II: Olsson’s approach.
- McKay numbers for symmetric groups.
- (If time allows) Block theory for symmetric groups.

Prerequisites

Prerequisites are minimal. Undergraduate representation theory (semisimplicity of the complex group algebra and first basic properties of complex characters), permutation representations. Group theory (symmetric groups and their conjugacy classes).

We plan to devote the first 2 lectures of the course to review the necessary background topics in the complex representation theory of finite groups.

Literature

1. C. Greene, A. Nijenhuis, and H. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, Adv. in Math. 31 (1979), no. 1, 104109.


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.
Modular Representation Theory (L24)
Stuart Martin

Modular representation theory, the study of representations over fields of characteristic other than zero, was initiated by L.E. Dickson, who first coined the term ‘modular’ representations. The first really major developments came with Richard Brauer, who exploited this rich and virtually untapped area of mathematics. Brauer’s methods were mainly character-theoretic; one of his main goals was finding numerical constraints on the orders and internal structure of finite simple groups, and his methods were very well-suited in the case of the so-called small groups.

The next revolution in the theory came with Sandy Green, who considered the modules themselves. His techniques were completely different to those of Brauer, and his main goals lay in understanding the modules, rather than Brauer, who worked mostly with blocks and characters.

The intention of this course is to give a flavour of both old fashioned and modern techniques. So we’ll discuss Brauer characters, defect groups, blocks, decomposition numbers as well as projective and injective modules.

- Review of assumed material: group algebras, representations and modules, reducibility and decomposability, Maschke’s theorem, tenors and homs, Frobenius reciprocity, maximal and primitive ideals, Jacobson radical, complete reducibility, semisimple rings, Artin-Wedderburn structure theorem.
- Modular character theory: $p$-singular and $p$-regular elements, Brauer characters, Grothen-dieck groups.
- Irreducible, projective and injective modules: DVRs, $p$-modular systems, decomposition numbers and the decomposition matrix, counting irreducible modules.
- Projective indecomposable modules: the Higman criterion, primitive orthogonal idempotents, idempotent refinement.
- Block theory I: the socle and the head, Cartan matrix, lifting projectives, central idempotents.
- Central characters: the central character of an ordinary irreducible mod $p$ determines the block.
- Block theory II: defect groups, relative projectivity and blocks of defect 0.
- (if time) Brauer’s first main theorem: (statement of) the Alperin conjecture, cyclic defect theory, finite, tame and wild representation type, the Brauer homomorphism and the Brauer correspondence.

One or two sheets of examples will be provided backed up by one or two classes.

Desirable Previous Knowledge

Basic group theory; ordinary representations/character theory from the Part II and other Part III courses; Sylow theory for finite groups; commutative algebras from Part III course (rings, ideals, completions, local rings, primality); some categorical nonsense.

Introductory Reading

1. J.L. Alperin & Rowen B. Bell, Groups and representation theory (GTM 162, Springer 1995)
2. C.W. Curtis, Pioneers of representation theory: Frobenius, Burnside, Schur and Brauer (AMS 1999)
3. I.M. Isaacs, Character theory of finite groups (Dover reprint 1994)
Reading to complement course material

1. J.L. Alperin, Local representation theory (CUP 1986)
3. P. Landrock, Finite group algebras and their modules (CUP 1984)
4. P. Webb, A course in finite group representation theory (CUP 2016)
Algebraic Geometry

Algebraic Geometry (M24)

P.M.H. Wilson

This will be a basic course introducing the tools of modern algebraic geometry, and applying them to deduce (for instance) the Riemann–Roch theorem for smooth projective curves. The most relevant reference for the course is the book of Kempf.

Topics to be covered are sheaves, abstract varieties (over an algebraically closed field) and their properties, coherent sheaves, divisors, sheaf cohomology, differentials and the Riemann–Roch Theorem. I shall not introduce schemes, but the proofs I’ll give will be in such a style that there are natural extensions to the case of schemes.

Pre-requisites

Basic theory on rings and modules will be assumed. Students will find it helpful to have looked beforehand at the book on Commutative Algebra by Atiyah and MacDonald, and/or the elementary text by Reid on Algebraic Geometry.

Literature

Introductory Reading


Reading to complement course material


Additional support

Four examples sheets will be provided and four associated examples classes will be given. Written answers to each examples sheet will be available a couple of days before the associated examples class.

Complex Manifolds (L24)

Daniel Pomerleano

A preliminary outline of the course is as follows, but this will almost certainly be subject to change.

- Basic concepts of complex manifolds, holomorphic vector bundles, holomorphic tangent and cotangent bundles (for which corresponding concepts from the real smooth manifolds will be assumed). Canonical line bundles, normal bundle for a submanifold and the adjunction formula.
• Brief description of sheaf cohomology, with deduction of de Rham and Dolbeault cohomology for complex manifolds.
• Hermitian metrics, connections, curvature and Chern classes for complex vector bundles. Case of holomorphic vector bundles.
• Harmonic forms: the Hodge theorem and Serre duality (general results on elliptic operators will be assumed).
• Compact Kahler manifolds. Hodge and Lefschetz decompositions on cohomology, Kodaira - Nakano vanishing, Kodaira embedding theorem.

Pre-requisites

A knowledge of basic Differential Geometry from the Michaelmas term will be highly desirable.

Literature


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Positivity in Algebraic Geometry (L18)

Roberto Svaldi

This class aims at giving an introduction to the theory of divisors, linear systems and their positivity properties on projective algebraic varieties.
The first part of the class will be dedicated to introducing the basic notions and results regarding these objects and special attention will be devoted to discussing examples in the case of curves and surfaces.
In the second part, the course will cover classical results from the theory of divisors and linear systems and their applications to the study of the geometry of algebraic varieties.
If time allows and based on the interests of the participants, there are a number of more advanced topics that could possibly be covered: Reider’s Theorem for surfaces, geometry of linear systems on higher dimensional varieties, multiplier ideal sheaves and invariance of plurigenera, higher dimensional birational geometry.

Pre-requisites

The minimum requirement for those students wishing to enroll in this class is their knowledge of basic concepts from the Algebraic Geometry Part 3 course, i.e., roughly Chapters 2 and 3 of Hartshorne’s Algebraic Geometry.
Familiarity with the basic concepts of the geometry of algebraic varieties of dimension 1 and 2 - e.g., as covered in the preliminary sections of Chapters 4 and 5 of Hartshorne’s Algebraic Geometry - would be useful but will not be assumed – besides what was already covered in the Michaelmas lectures.
Students should have also some familiarity with concepts covered in the Algebraic Topology Part 3 course such as cohomology, duality and characteristic classes.
Literature


Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

**Shimura varieties, some examples (L24)**

*Non-Examinable (Graduate Level)*

Ian Grojnowski

Modular curves — spaces like the quotient of the upper half plane by $SL_2(\mathbb{Z})$ — are ubiquitous in number theory and geometry. They paramaterise elliptic curves with level structure, are defined over number fields, and are a source of Galois representations.

Shimura varieties are generalisations of modular curves. They are an immensely rich class of examples which are of significance in geometry, where they parameterise certain Hodge structures, and in number theory, where they are a source of certain Galois representations.

This course will be cover some examples and basic theorems about Shimura varieties. Some of the results will be about arbitrary locally symmetric varieties.

I am really not an expert on this, and I'm not yet sure what theorems we will cover. Some topics we may include are: Tamagawa number theorems, complex multiplication, Gross-Kohnen-Zagier formulas, explicit automorphic forms, as well as examples of the relation to the Langlands program and to moduli problems in algebraic geometry.

Most of the theorems are already substantial and deep in the modular curve case.

Pre-requisites

This course assumes you know nothing but can do anything. The Part II Number Fields course, algebraic geometry, and the basics of Lie algebras will all be helpful, but aren’t required.

If you already have an opinion on Shimura varieties, global or local, you probably know more about these matters than I do. Please suggest essential theorems we need to cover, as well as telling me how I am doing it wrong,... (and do come along to class occasionally, and contribute).

Literature

There is an immense literature on Shumura varieties. We will slowly pick our way through parts of it.
Analysis

Analysis of Partial Differential Equations (M24)

Dr Warnick

This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on four prototype linear equations: Laplace’s equation, the heat equation, the wave equation and Schrödinger’s equation. Emphasis will be given to modern functional analytic techniques, relying on a priori estimates, rather than explicit solutions, although the interaction with classical methods (such as the fundamental solution and Fourier representation) will be discussed. The following basic unifying concepts will be studied: well-posedness, energy estimates, elliptic regularity, characteristics, propagation of singularities, group velocity, and the maximum principle. Some non-linear equations may also be discussed. The course will end with a discussion of major open problems in PDEs.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Preliminary Reading

The following article gives an overview of the field of PDEs:


Literature

1. Some lecture notes from a previous lecturer of the course are available online at: [http://cmouhot.wordpress.com/teachings/](http://cmouhot.wordpress.com/teachings/).

The following textbooks are excellent references:


Additional Information

This course is also intended for doctoral students of the Centre for Analysis (CCA), who will also be involved in additional assignments, presentations and group work. Part III students do not do these, and they will be assessed in the usual way by exam at the end of the academic year. Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be one office hour a week.
Functional Analysis (M24)

András Zsák

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. We will cover the following topics:

Hahn–Banach Theorem on the extension of linear functionals. Locally convex spaces.

Duals of the spaces $L_p(\mu)$ and $C(K)$. The Radon–Nikodym Theorem and the Riesz Representation Theorem.

Weak and weak-* topologies. Theorems of Mazur, Goldstine, Banach–Alaoglu. Reflexivity and local reflexivity.


Some additional topics time permitting. For example, the Fréchet–Kolmogorov Theorem, weakly compact subsets of $L_1(\mu)$, the Eberlein–Šmulian and the Krein–Šmulian theorems, the Gelfand–Naimark–Segal construction.

Pre-requisites

Thorough grounding in basic topology and analysis. Some knowledge of basic functional analysis and basic measure theory (much of which will be recalled either in lectures or via handouts). In Spectral Theory we will make use of basic complex analysis. For example, Cauchy’s Theorem, Cauchy’s Integral Formula and the Maximum Modulus Principle.

Literature


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be some material as well as examples sheets and announcements available at www.dpmms.cam.ac.uk/~az10000/

Topics in Ergodic Theory (M24)

Péter Varjú

Ergodic theory studies dynamical systems that are endowed with an invariant measure. There are many examples of such systems that originate from other branches of mathematics. This led to a fruitful interplay between ergodic theory and other fields, especially number theory.
I will explain some basic elements of ergodic theory, such as recurrence, ergodic theorems, mixing properties and entropy. I will also talk about some applications of the theory, such as Furstenberg’s proof of Szemerédi’s theorem, and Weyl’s equidistribution theorem for polynomials.

I aim to cover the following topics:

- Furstenberg’s correspondence principle,
- Poincaré recurrence, ergodicity,
- ergodic theorems,
- unique ergodicity,
- Weyl’s equidistribution theorem for polynomials,
- mixing and weak mixing,
- the multiple recurrence theorem for weak mixing systems,
- entropy and its relation to mixing,
- Rudolph’s theorem on $\times 2, \times 3$ invariant measures.

**Pre-requisites**

Measure theory, basic functional analysis, conditional expectation, Fourier transform.

**Literature**

Notes will be available on the lecturer’s webpage.

### Elliptic Partial Differential Equations (L24)

**Neshan Wickramasekera**

This course is intended as an introduction to the theory of linear second order elliptic partial differential equations. Second order elliptic equations play a fundamental role in many areas of mathematics including geometric analysis and mathematical physics. A strong background in the linear theory provides a foundation for studying a number of non-linear problems including minimal submanifolds, harmonic maps, geometric flows and general relativity. We will discuss both classical and weak solutions to linear elliptic equations focusing on the question of existence and uniqueness of solutions to the Dirichlet problem and the question of regularity of solutions. This involves establishing maximum principles, Schauder estimates and other a priori estimates for the solutions. As time permits, we will discuss other topics including the De Giorgi–Nash–Moser theory (which provides the Harnack inequality and establishes Hölder continuity for weak solutions) and applications of the linear theory to quasilinear elliptic equations.

**Pre-requisites**

Lebesgue integration, Lebesgue spaces, Sobolev spaces and basic functional analysis.

**Literature**

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.
Combinatorics (M16)

Béla Bollobás

What can one say about a collection of subsets of a finite set satisfying certain conditions in terms of containment, intersection and union? In the past fifty years or so, a good many fundamental results have been proved about such questions: in the course we shall present a selection of these results and their applications, with emphasis on the use of algebraic and probabilistic arguments.

The topics to be covered are likely to include the following.

The de Bruijn–Erdős theorem and its extensions.


The theorems of Sperner, EKR, LYMB, Katona, Frankl and Füredi.

Isoperimetric inequalities: Kruskal–Katona, Harper, Bernstein, BTBT, and their applications.

Correlation inequalities, including those of Harris, van den Berg and Kesten, and the Four Functions Inequality.

Alon’s Combinatorial Nullstellensatz and its applications.

LLL and its applications.

Pre-requisites

The main requirement is mathematical maturity, but familiarity with the basic graph theory course in Part II would be helpful.

Additional support

Almost all the material in the course will be self-contained; some parts of the course will be supported by printed notes. There will be three examples sheets and three associated examples classes. In addition, we shall have a one-hour revision class in the Easter Term.

Extremal Graph Theory (M16)

Andrew Thomason

Turán’s theorem, giving the maximum size of a graph that contains no complete $r$-vertex subgraph, is an example of an extremal graph theorem. Extremal graph theory is an umbrella title for the study of how graph and hypergraph properties depend on the values of parameters. This course builds on the material introduced in the Part II Graph Theory course, which includes Turán’s theorem and also the Erdős-Stone theorem.

The first few lectures will cover the Erdős-Stone theorem and stability. Then we shall treat Szemerédi’s Regularity Lemma, with some applications, such as to hereditary properties. Subsequent material, depending on available time, might include: hypergraph extensions, the flag algebra method of Razborov, graph containers and applications.
Pre-requisites

A knowledge of the basic concepts, techniques and results of graph theory, as afforded by the Part II Graph Theory course, will be assumed. This includes Turán’s theorem, Ramsey’s theorem, Hall’s theorem and so on, together with applications of elementary probability.

Literature

No book covers the course but the following can be helpful.


Additional support

Example sheets will be supplied, and three hours of examples classes will be given. There will be a further revision class in the Easter Term.

**Ramsey Theory (L16)**

Prof. I.B. Leader

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. A typical example is van der Waerden’s theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions.

The flavour of the course is combinatorial. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. We shall cover a number of ‘classical’ Ramsey theorems, such as Gallai’s theorem and the Hales-Jewett theorem, as well as some more recent developments. There will also be several indications of open problems.

We hope to cover the following material.

**Monochromatic Systems**


**Partition Regular Equations**

Definitions and examples. The columns property; Rado’s theorem. Applications. \((m,p,c)\)-sets and Deuber’s theorem. Ultrafilters; the Stone-Čech compactification. Idempotent ultrafilters and Hindman’s theorem.

**Infinite Ramsey Theory**

Basic definitions. Not all sets are Ramsey. Open sets and the Galvin-Prikry lemma. Borel sets are Ramsey. Applications.
Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey’s theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness.

Appropriate books


**Topics in Random Graphs (L16)**

*Non-Examinable (Graduate Level)*

Richard Montgomery

Random Graphs have been a fundamental object of study in combinatorics since the pioneering work of Erdős and Rényi in the 1960’s. Not only do random graphs have many interesting properties, but their study provides a rich space in which to mix combinatorial and probabilistic ideas. In this course, we will explore some of these ideas.

For any given (monotone) graph property (such as connectivity), if we steadily increase the density of edges in a random graph then at a certain point - known as the threshold for the property - the random graph will suddenly become very likely to have this property. We will determine the thresholds of a range of different graph properties, and also prove various hitting time results, where a complex property is shown to almost surely hold in a random graph if (and only if) some simpler property holds. The material covered will include coupling probability models, sprinkling edges, conditioning arguments, Pósa’s rotation technique, vertex absorption, and other methods.

**Pre-requisites**

We will assume only some very basic notions of probability and graph theory.

**Literature**

Algebraic Topology assigns algebraic invariants (groups and homomorphisms) to topological spaces and continuous maps between them. The most important example of such an invariant is ordinary homology theory, which is part of the basic language of geometry today. This course will cover homology and cohomology, together with applications to the topology of manifolds and vector bundles. The emphasis will be on learning to compute and use these invariants in a variety of examples. A tentative syllabus is as follows:

- **Vector Bundles.** Vector bundles and principal bundles. The Thom isomorphism and the Euler class. Long exact sequence on homotopy groups.
- **Topology of Manifolds.** Handle decompositions and Morse theory. Poincaré duality. The Lefshetz fixed point theorem.

**Pre-requisites**

The only required background is basic point-set topology, but prior experience with the fundamental group would be helpful. The material in the Michaelmas term Differential Geometry course will be useful as well.

**Literature**

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Differential Geometry (M24)
Ivan Smith

Differential geometry is the study of calculus on manifolds. This course is intended as an introduction to modern differential geometry, and should be useful to those interested in Geometry and Topology broadly-interpreted; the material is also relevant to parts of analysis and mathematical physics.

A tentative syllabus is as follows (depending on time, some more advanced topics may also be covered).

- **Basic theory**: Smooth manifolds and smooth maps. Tangent vectors and vector fields; the tangent bundle and the exponential map. Lie groups and Lie derivative.


- **Curvature**: Connections and covariant derivatives. Curvature, Chern-Weil theory. Torsion, the Levi-Civita connection. Riemannian manifolds.

Pre-requisites

An essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and multivariate calculus (e.g. differentiation and Taylor’s theorem in several variables). Exposure to some of the ideas of classical differential geometry would also be useful.

Literature


Additional support

Three or four examples sheets will be provided and four associated examples classes will be given. The fourth class will take place at the start of the Lent Term and will also serve a revision function.

Symplectic Geometry (L24)
Ana Rita Pires

The first part of the course will be an overview of the basic structures of symplectic geometry, including symplectic linear algebra, symplectic manifolds, symplectomorphisms, Darboux theorem, cotangent bundles, Lagrangian submanifolds, and Hamiltonian systems. The course will then go further into two topics. The first one is moment maps and toric symplectic manifolds, and the second one is capacities and symplectic embedding problems.
Pre-requisites

Some familiarity with basic notions from Differential Geometry and Algebraic Topology will be assumed. The material covered in the respective Michaelmas Term courses would be more than enough background.

Literature

Further references to survey papers will be provided as needed during the course.


Additional support

Three or four examples sheets will be provided and associated examples classes will be given.

3-manifolds (L24)

Sarah Rasmussen

This course aims to provide a survey of topics relevant to research in 3-manifold topology and geometry.

- **Knots and links.** Invariants of knots and links, including the Jones and Alexander polynomials. Categorification of invariants. Dehn filling and Dehn surgery.

- **Geometrization and Hyperbolic geometry.** A survey, mostly without proof, of primary notions from geometrization—which characterizes 3-manifold geometry and topology in terms of the fundamental group—and hyperbolic geometry.

- **3-manifold constructions.** Mapping tori, handle decompositions, Heegaard splittings, triangulations.

- **Foliations.** Singular, Reeb, and taut foliations, with connections to fundamental group actions, topology, and geometry. Transverse foliations on Seifert fibred spaces.

Pre-requisites


Literature


Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.
Characteristic classes and $K$-theory (L16)
Oscar Randal-Williams

This is an advanced course on the algebraic topology of vector bundles. In the first part of the course we will describe the basic constructions on vector bundles, and prove the classification of vector bundles by Grassmannians. We will then describe cohomological invariants of vector bundles, the Chern, Stiefel–Whitney, and Pontrjagin characteristic classes, establish their basic properties and give applications.

In the second part of the course we will define complex $K$-theory, a topological invariant analogous to cohomology but defined in terms of vector bundles rather than cochains. We will prove the fundamental Bott Periodicity Theorem, establish the basic properties of $K$-theory such as long exact sequences, representability, and the Thom isomorphism, then move on to operations on $K$-theory, applications (the celebrated Hopf Invariant One Theorem, counting vector fields on spheres), and relations between $K$-theory and cohomology.

Pre-requisites
Part III Algebraic Topology is essential.

Literature
   https://www.math.cornell.edu/~hatcher/VBKT/VB.pdf

Additional support
Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Geometric group theory (M24)
Non-Examinable (Graduate Level)
Henry Wilton

The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts. This graduate course will provide an introduction to the basic ideas of the subject.

Suppose $\Gamma$ is a discrete group of isometries of a metric space $X$. We focus on the theorems we can prove about $\Gamma$ by imposing geometric conditions on $X$. These conditions are motivated by curvature conditions in differential geometry, but apply to general metric spaces and are much easier to state. First we study the case when $X$ is Gromov-hyperbolic, which corresponds to negative curvature. Then we study the case when $X$ is $CAT(0)$, which corresponds to non-positive curvature. In order for this theory to be useful, we
need a rich supply of negatively and non-positively curved spaces. We develop the theory of *non-positively curved cube complexes*, which provide many examples of CAT(0) spaces and have been the source of some dramatic developments in low-dimensional topology over the last twenty years.

Part 1. We will introduce the basic notions of geometric group theory: Cayley graphs, quasi-isometries, the Schwarz–Milnor Lemma, and the connection with algebraic topology via presentation complexes. We will discuss the word problem, which is quantified using the Dehn functions of a group.

Part 2. We will cover the basic theory of word-hyperbolic groups, including the Morse lemma, local characterization of quasigeodesics, linear isoperimetric inequality, finitely presentedness, quasiconvex subgroups etc.

Part 3. We will cover the basic theory of CAT(0) spaces, working up to the Cartan–Hadamard theorem and Gromov’s Link Condition. These two results together enable us to check whether the universal cover of a complex admits a CAT(0) metric.

Part 4. We will introduce cube complexes, in which Gromov’s link condition becomes purely combinatorial. If there is time, we will discuss Haglund–Wise’s *special* cube complexes, which combine the good geometric properties of CAT(0) spaces with some strong algebraic and topological properties.

**Pre-requisites**

Part IB Geometry and Part II Algebraic topology are required.

**Literature**


**Additional support**

As a graduate course, none of this material is examinable. Nevertheless, the four example sheets from a previous Part III version of this course may provide a useful resource.
Logic

Logic (M24)

Thomas Forster

This course is the sequel to the Part II courses in Set Theory and Logic and in Automata and Formal Languages lectured in 2015-6. (It is already being referred to informally as “Son of ST& L and Automata & Formal Languages”). Because of the advent of that second course this Part III course no longer covers elementary computability in the way that its predecessor (“Computability and Logic”) did, and this is reflected in the change in title. It will say less about Set Theory than one would expect from a course entitled ‘Logic’; this is because in Lent term Benedikt Löwe will be lecturing a course entitled ‘Topics in Set Theory’ and I do not wish to tread on his toes. Material likely to be covered include: advanced topics in first-order logic (Natural Deduction, Sequent Calculus, Cut-elimination, Interpolation, Skolemisation, Completeness and Undecidability of First-Order Logic, Curry-Howard, Possible world semantics, Gödel’s Negative Interpretation, Generalised quantifiers . . . ); Advanced Computability (\(\lambda\)-representability of computable functions, Tennenbaum’s theorem, Friedberg-Muchnik, Baker-Gill-Solovay . . . ); Model theory background (ultraproducts, Lö’s theorem, elementary embeddings, omitting types, categoricity, saturation, Ehrenfeucht-Mostowski theorem . . . ); Logical combinatorics (Paris-Harrington, WQO and BQO theory at least as far as Kruskal’s theorem on wellquasiorderings of trees . . . ).

Pre-requisite Mathematics

The obvious prerequisites from last year’s Part II are my Set Theory and Logic and Dr Chiodo’s Automata and Formal Languages, and I would like to assume that everybody coming to my lectures is on top of all the material lectured in those courses. Attending these two Part II courses in Michaelmas is a course of action that may appeal particularly to students from outside Cambridge.

Literature

J.L. Bell and Alan Slomson Models and Ultraproducts. Dover
T. E. Forster: Logic, Induction and Sets CUP.
(Errata are on http://www.dpmms.cam.ac.uk/~tf/typoslis.html)
P. T. Johnstone: Notes on Set theory CUP.
Wilfrid Hodges: Model theory CUP (long and short versions).

Teaching materials will be linked from the page on http://www.dpmms.cam.ac.uk/~tf/partiii.html

Category Theory (L24)

Prof. P.T. Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just
a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

**Categories, functors and natural transformations.** Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]

**Locally small categories.** The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]

**Adjunctions.** Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]

**Limits.** Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]

**Monads.** The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck’s Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

**Filtered colimits.** Finitary functors, finitely-presentable objects. Applications to universal algebra.

**Regular categories.** Embedding theorems. Categories of relations, introduction to allegories.

**Abelian categories.** Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.

**Monoidal categories.** Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

**Fibrations.** Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

**Pre-requisites**

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

**Literature**

2. S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane’s classic (Awodey was Mac Lane’s last PhD student), but rather more gently paced.
3. T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn’t cover the whole course.
4. F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

**Additional support**

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.
This course covers advanced topics in set theory, focusing on meta-mathematical techniques such as inner models and forcing.

Set theory and logic are intrinsically intertwined since the most interesting results in set theory are independence results showing that natural questions in set theory are not solvable using the standard axiomatic system of Zermelo-Fraenkel set theory with choice ZFC.

The most famous of these natural questions is Cantor’s continuum hypothesis $\text{CH}$, “every uncountable set of reals is equinumerous to the set of all real numbers” or, equivalently, $2^{\aleph_0} = \aleph_1$. This question was elevated to the status of the foremost mathematical problem for the 20th century by David Hilbert in his address to the International Congress of Mathematicians in Paris in the year 1900. In 1938, Kurt Gödel proved that $\text{CH}$ cannot be disproved in ZFC (inventing and using the method of inner models); in 1963, Paul Cohen proved that $\text{CH}$ cannot be proved in ZFC (inventing and using the method of forcing). Together, these results show that $\text{CH}$ is independent from ZFC.

We shall treat several of the following topics:


**Large cardinals.** Introduction to large cardinals. Inaccessible cardinals. Measurable cardinals. Ultra-powers. Scott’s theorem.

**Inner models.** Definability. Ordinal definability. Constructibility. Condensation. Gödel’s proof of the consistency of $\text{CH}$.

**Forcing.** Generic extensions. The forcing theorems. Adding reals; collapsing cardinals. Cohen’s proof of the consistency of $\neg\text{CH}$.

**Pre-requisites**

The Part II course *Logic and Set Theory* or an equivalent course is essential.

**Additional support**

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.
Number Theory

Algebraic Number Theory (M24)
Christian Johansson

Algebraic number theory is one of the cornerstones of modern number theory. It provides important tools for the study of diophantine equations (such as Fermat’s Last Theorem) and other areas, and is a beautiful subject in its own right. This will be a second course in algebraic number theory, focusing on local ($p$-adic) and Galois theoretic aspects of the subject.

Topics likely to be covered include:

- Dedekind domains and discrete valuation rings. Global and local fields, the $p$-adic numbers.
- Finite extensions, Galois theory and ramification theory.
- Adèles and idèles, the idèle class group.
- Statements of Local and Global Class field theory, some applications.

Pre-requisites

Algebra, at the level of Part II Galois Theory and Part IB Groups, Rings and Modules, is essential. Prior exposure to algebraic number theory, at the level of Part II Number Fields, will be assumed but might not be logically necessary to follow the course.

Literature


Additional support

There will be four examples sheets, with four associated examples classes. There will also be a revision session in the Easter term.

Modular Forms and $L$-Functions (M24)
Prof. A. J. Scholl

Modular Forms are classical objects that appear in many areas of mathematics, from number theory to representation theory and mathematical physics. Most famous is, of course, the role they played in the proof of Fermat’s Last Theorem, through the conjecture of Shimura-Taniyama-Weil that elliptic curves are modular. One connection between modular forms and arithmetic is through the medium of $L$-functions, the basic example of which is the Riemann $\zeta$-function. We will discuss various types of $L$-function in this course and give arithmetic applications.

Pre-requisite Mathematics
Prerequisites for the course are fairly modest; from number theory, apart from basic elementary notions, some knowledge of quadratic fields is desirable. A fair chunk of the course will involve (fairly 19th-century) analysis, so we will assume the basic theory of holomorphic functions in one complex variable, such as are found in a first course on complex analysis (e.g. the 2nd year Complex Analysis course of the Tripos).

Books


3. F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Maths. 228, Springer, New York, 2005 (a good reference providing also an introduction to the algebraic theory of modular forms, although goes into a lot more detail than we will give in this course).


Additional references for enthusiasts


Topics in Number Theory (L16)

*Non-Examinable (Graduate Level)*

Prof. A. J. Scholl

The “Langlands programme” is a far-ranging series of conjectures describing the connections between automorphic forms on the one hand, and algebraic number theory and arithmetic algebraic geometry on the other. In these lectures we will give an introduction to some aspects of this programme.

Pre-requisite Mathematics

The course will follow on naturally from the Michaelmas term courses *Algebraic Number Theory* and *Modular Forms and $L$-Functions*, and knowledge of them will be assumed. Some knowledge of algebraic geometry will be required in places.

Relevant literature:


2. J. Bernstein, S. Gelbart (eds.), *An Introduction to the Langlands Program*. Birkhäuser, 2004
The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

**Review of measure and integration:** sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini’s theorem.

**Conditional expectation:** Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.

**Martingales:** Martingales and submartingales in discrete time; optional stopping; Doob’s inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.

**Stochastic processes in continuous time:** Kolmogorov’s criterion, regularization of paths; martingales in continuous time.

**Weak convergence:** Definitions and characterizations; convergence in distribution, tightness, Prokhorov’s theorem; characteristic functions, Lévy’s continuity theorem.

**Sums of independent random variables:** Strong laws of large numbers; central limit theorem; Crâmer’s theory of large deviations.

**Brownian motion:** Wiener’s existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker’s invariance principle.

**Poisson random measures:** Construction and properties; integrals.

**Lévy processes:** Lévy-Khinchin theorem.

**Pre-requisites**

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams’ book) to strengthen their understanding.

**Literature**

- Lecture notes online: [www.statslab.cam.ac.uk/~james/Lectures/ap.pdf](http://www.statslab.cam.ac.uk/~james/Lectures/ap.pdf)
- D. Williams, Probability with martingales, CUP 1991.

**Additional support**

Four example sheets will be provided along with supervisions. There will be a revision class in Easter term.
Percolation and Random Walks on Graphs (M16)

Perla Sousi

A phase transition means that a system undergoes a radical change when a continuous parameter passes through a critical value. We encounter such a transition every day when we boil water. The simplest mathematical model for phase transition is percolation. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for a solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm-Loewner evolutions (SLE), and to other models from statistical physics. The basic theory of percolation will be described in this course with some emphasis on areas for future development.

Our other major topic includes random walks on graphs and their intimate connection to electrical networks; the resulting discrete potential theory has strong connections with classical potential theory. We will develop tools to determine transience and recurrence of random walks on infinite graphs. Other topics include the study of spanning trees of connected graphs. We will present two remarkable algorithms to generate a uniform spanning tree (UST) in a finite graph G via random walks, one due to Aldous-Broder and another due to Wilson. These algorithms can be used to prove an important property of uniform spanning trees discovered by Kirchhoff in the 19th century: the probability that an edge is contained in the UST of G, equals the effective resistance between the endpoints of that edge.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature


Advanced Financial Models (M24)

M.R. Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability and Stochastic Calculus & Applications.

- **Discrete time models.** Filtrations and martingales. Arbitrage, martingale deflators and equivalent martingale measures. Attainable claims and market completeness. European and American claims. Optimal stopping.
- **Interest rate models.** Short rates, forward rates and bond prices. Markovian short rate models. The Heath–Jarrow–Morton drift condition.
Pre-requisites

Familiarity with measure-theoretic probability will be assumed.

Literature


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Stochastic Networks (L16)

Frank Kelly

Communication networks underpin our modern world, and provide fascinating and challenging examples of large-scale stochastic systems. This course uses stochastic models to shed light on important issues in the design and control of communication networks.

Randomness arises in communication systems at many levels: for example, the initiation and termination times of calls in a telephone network, or the statistical structure of the arrival streams of packets at routers in the Internet. How can routing, flow control and connection acceptance algorithms be designed to work well in uncertain and random environments? And can we design these algorithms using simple local rules so that they produce coherent and purposeful behaviour at the macroscopic level?

The course will start with a variety of classical models that can be used to help understand the performance of large-scale stochastic networks. Queueing and loss networks will be studied, as well as random access schemes and the concept of an effective bandwidth. Parallels will be drawn with models from physics, and with models of traffic in road networks. Next the course will describe models of packet traffic and of congestion control algorithms in the Internet. The complex interplay between end-systems and the network has attracted the attention of economists as well as mathematicians and engineers.

We describe enough of the technological background to communication networks to motivate our models, but no more. Some of the ideas described in the book are finding application in financial, energy, and economic networks as computing and communication technologies transform these areas. But communication networks currently provide the richest and best developed area of application within which to present a connected account of the ideas.

Pre-requisites

Mathematics that will be assumed to be known before the start of the course: Part IB Optimization and Markov Chains. Familiarity with Part II Applied Probability would be useful, but is not assumed.
Preliminary Reading


Literature

1. B. Hajek *Communication Network Analysis*.
3. F. Kelly and E. Yudovina *Stochastic Networks*. Cambridge University Press, 2014. (The course text.)

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

**Stochastic Calculus and Applications (L24)**

R. Bauerschmidt

This course will be an introduction to Itô calculus.

- **Brownian motion.** Existence and sample path properties.
- **Stochastic calculus for continuous processes.** Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô’s isometry, definition of the stochastic integral, Kunita-Watanabe theorem, and Itô’s formula.
- **Applications to Brownian motion and martingales.** Lévy characterization of Brownian motion, Dubins-Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- **Stochastic differential equations.** Strong and weak solutions, notions of existence and uniqueness, Yamada-Watanabe theorem, strong Markov property, and relation to second order partial differential equations.
- **Stroock–Varadhan theory.** Diffusions, martingale problems, equivalence with SDEs, approximations of diffusions by Markov chains.

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with discrete-time martingales and Brownian motion.
Schramm-Loewner Evolutions (L16)

J. P. Miller

Schramm-Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter \( \kappa \geq 0 \). These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for \( \kappa \leq 4 \) the curves are simple, for \( 4 \leq \kappa < 8 \) they have double points but are non-crossing, and for \( \kappa \geq 8 \) they are space-filling. We will then explore the properties of the curves for a number of special values of \( \kappa \) (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

Additional support

Two examples sheets will be provided and examples classes given. There will be a revision class in Easter Term.
Statistics and Operational Research

The courses in statistics form a coherent Masters-level course in statistics, covering statistical methodology, theory and applications. You may take all of them, or a subset of them. Core courses are Modern Statistical Methods and Applied Statistics in the Michaelmas Term.

All statistics courses for examination in Part III assume that you have taken an introductory course in statistics and one in probability, with syllabuses that cover the topics in the Cambridge undergraduate courses Probability in the first year and Statistics in the second year. It is helpful if you have taken more advanced courses, although not essential. For Applied Statistics and other applications courses, it is helpful, but not essential, if you have already had experience of using a software package, such as R or Matlab, to analyse data. The statistics courses assume some mathematical maturity in terms of knowledge of basic linear algebra and analysis. However, they are designed to be taken without a background in measure theory, although some knowledge of measure theory is helpful for Topics in Statistical Theory.

The desirable previous knowledge for tackling the statistics courses in Part III is covered by the following Cambridge undergraduate courses. The syllabuses are available online at

https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf

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<td>First</td>
<td>Essential Probability</td>
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<td>Second</td>
<td>Essential Statistics</td>
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<td>Third</td>
<td>Helpful Principles of Statistics</td>
<td>Statistical Modelling</td>
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<td>Probability and Measure</td>
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If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation. If you have more time, then it would be helpful to review other courses as indicated.

Modern Statistical Methods (M24)

Rajen Shah

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data where we have huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to analyse such datasets. We aim to cover some of the following topics.

- Kernel machines: the kernel trick, the representer theorem, support vector machines, the hashing trick.
- Penalised regression: Ridge regression, the Lasso and variants.
• Graphical modelling: neighbourhood selection and the graphical Lasso. Causal inference through structural equation modelling; the PC algorithm.

• High-dimensional inference: the closed testing procedure and the Benjamini–Hochberg procedure; the debiased Lasso.

Pre-requisites

Basic knowledge of statistics, probability, linear algebra and real analysis. Some background in optimisation would be helpful but is not essential.

Literature


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

**Topics in Statistical Theory (M16)**

**Thomas B. Berrett**

This course will provide an introduction to the theory behind a selection of statistical problems that play a key role in modern statistics. Most undergraduate statistics courses are restricted to the study of parametric models; here we will no longer assume that our distributions belong to finite dimensional classes and will instead study fundamental nonparametric problems such the estimation of a distribution function, a density function or a regression function. We will consider the canonical machine learning problem of classification, and may also cover some extreme value theory including analogues of the Central Limit Theorem for the maxima and minima of a sample. Minimax lower bounds are studied as a way of quantifying the intrinsic difficulty of a statistical problem, and provide limits on how well any estimator can perform in a given situation.

A tentative outline of the course is as follows:


- Kernel density estimation: histograms, bias and variance expansions, asymptotically optimal bandwidth, canonical kernels, higher order kernels, bandwidth selection, multivariate density estimation.

- Nonparametric regression: kernel nonparametric regression, bias and variance expansions. Cubic splines, natural cubic smoothing splines, choice of smoothing parameter, other splines, equivalent kernel. Classification problems, the Bayes classifier, nearest neighbour classifiers.

- Minimax theory: notion of information-theoretic lower bounds, distance and divergence between distributions, optimal rates, Le Cam’s two points lemma.

- Extreme value theory: the extremal types theorem, domains of attraction, max-stability.
Pre-requisites

A good background in undergraduate probability theory, though measure theory is not necessary; elements of linear algebra; a preliminary course in mathematical statistics can be helpful, but is not necessary. Though the material in the Modern Statistical Methods course will not be needed here, the two courses complement each other well.

Literature

No book will be explicitly followed, but some of the material is covered in

Additional support

Three example sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Statistics in Medicine (24 units)

This course consists of two components, Statistics in Medical Practice (12 lectures) and Analysis of Survival Data (12 lectures). Together these make up one 3 unit (24 lecture) course. You must take the two components together for the examination.

Statistics in Medical Practice (M12)

(Lecturers from the MRC Biostatistics Unit)

This part of the course includes three modules covering a range of statistical methods and their application in three areas of biostatistics.

A. Stochastic Models for Chronic and Infectious Diseases [4 Lectures] (C. Jackson, A. Presanis & P. Birrell)


B. Design and Analysis of Randomised Trials [4 Lectures] (J. Wason, S. Villar & S. Seaman)


C. Genetics and Genomics [4 Lectures] (S. Burgess, P. Kirk & S. Hill)

Introduction to Mendelian randomization. Instrumental variables and natural experiments. Practical examples of how genetics can be used to make causal inferences and guide target selection for improving disease outcomes. Some recent theoretical developments on robust causal inferences with

Analysis of Survival Data (L12)

(P. Treasure, Department of Public Health and Primary Care)

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

D. Time-to-Event Analysis [4 Lectures]

'Survival analysis' is generalised to time-to-event analysis. The implications of event times which are unknown or in the future (censored data) are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (Kaplan-Meier) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (log-rank test).

E. Modelling Hazard [4 Lectures]

The hazard function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size ... ) on the time-to-event distribution (proportional hazards modelling). Model checking procedures are introduced with an emphasis on excess event (Martingale) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (competing risk analysis) and (ii) the cause of death is unknown (net survival). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar time (e.g. 2017) are addressed: period survival analysis.

Pre-requisites

Undergraduate-level statistical theory, including analysis and interpretation of data, maximum likelihood estimation and hypothesis testing.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. A few books to complement the course material are listed below.


**Additional support**

Example and revision classes will be given, with question sheets and solutions.

**Bayesian Modelling and Computation (L24)**

Sergio Bacallado

The course will cover a range of algorithms for sampling and numerical integration which are useful in Bayesian inference. A third of the lectures will deal with applications in specific statistical models.

- **Fundamentals**: Monte Carlo integration; variance reduction techniques; rejection sampling and adaptive rejection sampling; importance sampling. Exponential families and conjugate priors.

- **Graphical models**: Belief networks, Markov random fields, and factor graphs; Hammersley–Clifford theorem; computational reduction between marginalisation, computing partition function, and sampling; belief propagation in trees.


- **Approximate inference**: Expectation maximisation. Variational inference; mean-field methods. Stochastic variational inference. The parametric Bootstrap and Bayesian methods.


**Pre-requisites**

This course assumes familiarity with probability and basic Markov chain theory. Knowledge of statistical modelling and Bayesian analysis is helpful.

**Literature**


Statistical Learning in Practice (L24)

Tengyao Wang

Statistical learning is the process of using data to guide the construction and selection of models, which are then used to predict future outcomes. In this course, which consist of 12 lectures and 12 practical classes, we will examine some of the most successful and widely used statistical methodologies in modern applications. The practical classes will deal with an introduction to R, exploratory data analysis and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models for regression and classification
- Model selection and regularisation
- Mixed effects models and quasi-likelihood methods
- Linear discriminant analysis and support vector machines
- Introduction to neural networks
- Time series and spatial statistics

Pre-requisites

Elementary probability theory. Maximum likelihood estimation, hypothesis tests and confidence intervals. Linear models.

Previous experience with R is helpful but not essential.

Literature


Additional support

This course includes practical classes, where statistical methods are introduced in a practical context and where students carry out analysis of datasets using R. In practical classes, the students have the opportunity to discuss statistical questions with the lecturer. Four examples sheets will be provided and there will be four associated examples classes. There will be a revision class in the Easter Term.
Astrostatistics (L24)

Kaisey Mandel

This course will cover statistical methods necessary to properly interpret today's increasingly complex datasets in astronomy. Particular emphasis will be placed on principled statistical modeling of astrophysical data and the statistical computation of inferences of scientific interest. Statistical problems and techniques, such as Bayesian modeling, nonparametric methods, density estimation, regression, classification, time series analysis, sampling methods, and machine learning, will be examined in the context of applications to modern astronomical data analysis. Examples will be drawn from applications across astrophysics and cosmology.

Literature


Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates $q$ and corresponding momenta $p$. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|j\ell m\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.


Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum $p^\mu$ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Basic knowledge of $\delta$-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation.

Statistical Field Theory (M16)

David Tong

This course is an introduction to the renormalization group, the basis for a modern understanding of field theory. The course is primarily focussed on statistical systems such as spin models, but the connection to quantum field theory is never far from the surface.

The simplest spin system – known as the Ising model – is a constant companion throughout the course. After a description of this model, Landau’s mean field theory is introduced and used as a framework
to discuss the phenomenology and classification of phase transitions. The extension to Landau-Ginzburg theory provides for a more complete understanding of fluctuations, and makes the connection to quantum field theory manifest.

These approaches struggle near a second order phase transition, also known as a "critical point". Here the tools of the renormalisation group become essential. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems.

**Pre-requisites**

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

**Literature**


**Additional support**

Three examples sheets will be provided and three associated examples classes will be given.

**Quantum Field Theory (M24)**

B. Allanach

Quantum Field Theory is the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using Lagrangian language and Noether’s theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics.

Interactions between fields are examined next, using the interaction picture, Dyson’s formula and Wick’s theorem. A ‘short version’ of these techniques is then introduced: Feynman diagrams.

Fermions and the Dirac equation are explored in detail, along with parity and $\gamma^5$. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions.

Finally, quantum electrodynamics is developed. A connection between the field strength tensor and Maxwell’s equations is carefully made, before gauge symmetry is introduced. Lorentz gauge is used as an example, before quantisation of the electromagnetic field and the Gupta-Bleuler condition. The interactions between photons and charged matter is governed by the principal of minimal coupling, which is covered next. A few examples of quantum amplitudes in QED are given.
**Symmetries, Fields and Particles (M24)**

N. Dorey

This course introduces the theory of Lie groups and Lie algebras and their applications to high energy physics. The course begins with a brief overview of the role of symmetry in physics. After reviewing basic notions of group theory we define a Lie group as a manifold with a compatible group structure. We give the abstract definition of a Lie algebra and show that every Lie group has an associated Lie algebra corresponding to the tangent space at the identity element. Examples arising from groups of orthogonal and unitary matrices are discussed. The case of SU(2), the group of rotations in three dimensions is studied in detail. We then study the representations of Lie groups and Lie algebras. We discuss reducibility and classify the finite dimensional, irreducible representations of SU(2) and introduce the tensor product of representations. The next part of the course develops the theory of complex simple Lie algebras. We define the Killing form on a Lie algebra. We introduce the Cartan-Weyl basis and discuss the properties of roots and weights of a Lie algebra. We cover the Cartan classification of simple Lie algebras in detail. We describe the finite dimensional, irreducible representations of simple Lie algebras, illustrating the general theory for the Lie algebra of SU(3). The last part of the course discusses some physical applications. After a general discussion of symmetry in quantum mechanical systems, we review the approximate SU(3) global symmetry of the strong interactions and its consequences for the observed spectrum of hadrons. We introduce gauge symmetry and construct a gauge-invariant Lagrangian for Yang-Mills theory coupled to matter. The course ends with a brief introduction to the Standard Model of particle physics.

**Pre-requisites**

Basic finite group theory, including subgroups and orbits. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

**Literature**

Additional support

A set of course notes will provided as handouts in the lectures. Printed notes of previous version of the course are also available on the Part III Examples and Lecture Notes webpage. Four examples sheets will be provided and four associated examples classes in moderate-sized groups will be given by graduate students.

Advanced Quantum Field Theory (L24)

DB Skinner

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalisation of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantising a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson Loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study Renormalization. Wilson’s picture of Renormalisation is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the Renormalisation Group (RG) flow. The course explores renormalisation systematically, from the use of dimensional regularisation in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as “asymptotic freedom”, this phenomenon revolutionised our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrise possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

Preliminary Reading


Literature


Additional support

There will be four problem sheets handed out during the course. Classes for each of these sheets will be arranged during Lent Term. There will also be a general revision class during Easter Term.

Standard Model (L24)

C.E. Thomas

The Standard Model of particle physics is, by far, the most successful application of quantum field theory (QFT). At the time of writing, it accurately describes all experimental measurements involving strong, weak, and electromagnetic interactions. The course aims to demonstrate how this model, a QFT with gauge group $SU(3) \times SU(2) \times U(1)$ and fermion fields for the leptons and quarks, is realised in nature. It is intended to complement the more general Advanced QFT course.

We begin by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content (spin-half leptons and quarks, spin-one gauge bosons and spin-zero Higgs boson). The parity $P$, charge-conjugation $C$ and time-reversal $T$ transformation properties of the theory are investigated. These need not be symmetries manifest in nature; e.g. only left-handed particles feel the weak force and so it violates parity symmetry. We show how $CP$ violation becomes possible when there are three generations of particles and describe its consequences.

Ideas of spontaneous symmetry breaking are applied to discuss Goldstone’s theorem, the Higgs mechanism and why the weakness of the weak force is due to the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry.

We show how to obtain cross sections and decay rates, quantities which can be measured in experiments, from the matrix element of a process. These can be computed for various scattering and decay processes in the electroweak sector using perturbation theory because the couplings are small. We touch upon the topic of neutrino masses and oscillations, an important window to physics beyond the Standard Model.

The strong interaction is described by quantum chromodynamics (QCD), the non-abelian gauge theory of the (unbroken) $SU(3)$ gauge symmetry. At low energies quarks are confined and form bound states called hadrons. The coupling constant decreases as the energy scale increases, to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. Time permitting, we may discuss nonperturbative approaches to QCD. For example, the framework of effective field theories can be used to make progress in the limits of very small and very large quark masses.

Both very high-energy experiments and very precise experiments are currently striving to observe effects that cannot be described by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model.

Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or to be familiar with the material covered in them. It would be advantageous to attend the Advanced QFT course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

Reading to complement course material

String Theory (L24)

Paul K. Townsend

String Theory supposes that elementary particles are excitations of a string, which could be open (with two endpoints) or closed. Closed strings have a massless spin-2 particle in their spectrum, which suggests that String Theory is a theory of quantum gravity. Open strings yield analogous generalisations of gauge theory, so String Theory can potentially unify gravity with the standard model of particle physics.

This course will introduce the strings of string theory as constrained Hamiltonian systems, focusing initially on the Nambu-Goto (NG) string, and using insights provided by a similar investigation of the relativistic point particle. Mention will be made of the relevance of NG strings to the “cosmic strings” encountered in cosmology.

Various methods of quantisation of the NG string, including light-cone gauge and “old covariant” (and possibly BRST) will be explained; this will reveal that there is a critical space-time dimension (26) and that the ground state is a tachyon. A study of the possible boundary conditions on open strings will suggest an interpretation in terms of branes.

Superstring theory will be introduced, in the RNS formalism (with possible mention of the GS formalism). The light-cone gauge will be used to show that the critical spacetime dimension is now 10. It will be explained briefly why superstring theories are tachyon-free and why there are five of them.

The path integral formulation of QM will be explained, and why you don’t need fields to do QFT. The generalisation to strings will lead to ideas of conformal field theory, a computation of the Virasoro-Shapiro amplitude for the scattering of closed-string tachyons of the NG string, and a discussion of some general features of string perturbation theory. This will include a look at the “one-loop” quantum corrections and why there are no UV divergences.

Other topics that may be discussed are T-duality and how the five superstring theories are unified by “M-Theory”.

Pre-requisites

This course assumes you know the basics of (i) Special Relativity and (ii) Quantum Mechanics. Complete typed course notes will be provided. The course structure is rather different from anything that can be found in text books or on-line reviews but the following are useful for general background and/or specific topics covered in the course:
Literature


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will also be a weekly office hour during the Lent term for questions about the lectures.

Classical and Quantum Solitons (E16)

N. S. Manton

Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantisation, they give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, and on Skyrmions in three dimensions. Solitons in gauge theories will also be mentioned.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Literature


Additional support

Two examples sheets will be provided and two associated examples classes will be given.

Supersymmetry (E16)

F. Quevedo

This course provides an introduction to the use of supersymmetry in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables and relates fermions and bosons.

Firstly, a physical motivation for supersymmetry is provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced. If time allows a short discussion of supersymmetry in higher dimensions will be briefly discussed.

Three examples sheets and examples classes will complement the course.
Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them.

Preliminary Reading


Literature

For more advanced topics later in the course, it will helpful to have a knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model in order to understand the final lecture on the minimal supersymmetric standard model.

Beware: most of the supersymmetry references contain errors in minus signs, aside (as far as I know) Wess and Bagger.

1. Course lecture notes from last year: [http://www.damtp.cam.ac.uk/user/examples/3P7.pdf](http://www.damtp.cam.ac.uk/user/examples/3P7.pdf)

2. Videos of a very similar lecture course: follow the links from [http://users.hepforge.org/~allanach/teaching.html](http://users.hepforge.org/~allanach/teaching.html)


4. Introduction to supersymmetry, J.D. Lykken, hep-th/9612114. This introduction is good for extended supersymmetry and more formal aspects.

5. Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this terse and more mathematical book has the opposite sign of metric to the course.

6. A supersymmetry primer, S.P. Martin, hep-ph/9709256 is good and detailed for phenomenological aspects, although with the opposite sign metric to the course.
Relativity and Gravitation

These courses provide a thorough introduction to General Relativity and Cosmology. The Michaelmas term courses introduce these subjects, which are then developed in more detail in the Lent term courses on Black Holes and Advanced Cosmology. A non-examinable course explores the application of spinor techniques in General Relativity.

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum $p^\mu$ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and $\delta$-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

https://www.maths.cam.ac.uk/undergrad/course

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<thead>
<tr>
<th>Year</th>
<th>Courses</th>
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<tbody>
<tr>
<td>First</td>
<td><strong>Essential:</strong> Vectors &amp; Matrices, Diff. Eq., Vector Calculus, Dynamics &amp; Relativity.</td>
</tr>
<tr>
<td>Second</td>
<td><strong>Essential:</strong> Methods, Quantum Mechanics, Variational Principles.</td>
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<tr>
<td></td>
<td><strong>Helpful:</strong> Electromagnetism, Geometry, Complex Methods.</td>
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<tr>
<td>Third</td>
<td><strong>Essential:</strong> Classical Dynamics.</td>
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<tr>
<td></td>
<td><strong>Helpful:</strong> Further Complex Methods, Asymptotic methods.</td>
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</table>

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation.

Cosmology (M24)

Paul Shellard

This course covers the last 13.8 billion years of the evolution of your universe, from the initial inflationary quantum perturbations to the creation of galaxies we observe today. The course will follow the following format

1. Geometry and Dynamics
2. Inflation
3. Cosmological Perturbation Theory
4. Structure Formation
5. Thermal History
6. Initial Conditions from Inflation
Pre-requisites

This course is taught in a self contained manner so could be attempted by any sufficiently keen part III student but some basic knowledge of Relativity, Quantum Mechanics and Statistical Mechanics will likely be quite helpful.

Literature

1. Dodelson, Modern Cosmology
2. Kolb and Turner, The Early Universe
3. Weinberg, Cosmology

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

General Relativity (M24)

M. Dunajski and H. Reall

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

Pre-requisites

This course will be self-contained, so previous knowledge of General Relativity is not essential. However, many students have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before, then it is strongly recommended that you study an introductory book (e.g. Hartle or Schutz) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Familiarity with Newtonian Gravity and special relativity is essential. Knowledge of the relativistic formulation of electrodynamics is desirable. Familiarity with finite-dimensional vector spaces, the calculus of functions $f : \mathbb{R}^m \to \mathbb{R}^n$, and the Euler-Lagrange equations will be assumed.

Preliminary Reading


Literature


This course is closest in content to Wald’s book. Stewart’s book gives a concise overview of the differential geometry in the first part of this course. Misner, Thorne and Wheeler’s book is particularly useful for the sections on gravitational radiation and the Newtonian limit. Carroll’s book is a very readable introduction.
Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Course website: [http://www.damtp.cam.ac.uk/user/hsr1000/teaching.html](http://www.damtp.cam.ac.uk/user/hsr1000/teaching.html)

**Black Holes (L24)**

J. Santos

A black hole is a region of spacetime that is causally disconnected from the rest of the Universe. These objects appear to be pervasive in Nature, and their properties have direct implications for the recent advances in gravitational wave astronomy. Besides being astrophysically relevant, black holes also play a fundamental role in quantum theory and are a natural arena to study and test any consistent quantum theory of gravity.

The following topics will be discussed:


2. The initial value problem, strong cosmic censorship.

3. Causal structure, null geodesic congruences, Penrose singularity theorem.

4. Penrose diagrams, asymptotic flatness, weak cosmic censorship.

5. Reissner-Nordström and Kerr black holes.


7. Positivity of energy theorem.

8. The laws of black hole mechanics. The analogy with laws of thermodynamics.


**Pre-requisites**

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

**Literature**


**Additional support**

Four examples sheets will be distributed during the course. Four examples classes will be held to discuss these. A revision class will be held in the Easter term.
Advanced Cosmology (L24)
Anthony Challinor and Tobias Baldauf

This course will take forward at much greater depth some of the topics in modern cosmology covered in the Michaelmas Term course. The prediction from fundamental theory for the statistical properties of the primordial perturbations remains the key area of confrontation with cosmological observations, both from large-scale structure and the cosmic microwave background (CMB). This course will develop the mathematical tools and physical understanding necessary for research in this very active area.

**Cosmic microwave background**
- Statistics of random fields
- Relativistic kinetic theory
- The Boltzmann equation
- The CMB temperature power spectrum
- Photon scattering and diffusion
- Primordial gravitational waves and the CMB
- CMB Polarization

**Inflationary theory and Large-Scale Structure**
- Primordial non-Gaussianities
- Effective field theory of inflation
- CMB bispectrum and optimal estimators
- Modelling late time non-linearities in large-scale structure
- Effective field theory of large-scale structure
- Tracers of large-scale structure and the peak formalism

**Pre-requisites**
Material from the Michaelmas term *Cosmology* is essential. Familiarity with introductory Quantum Field Theory is recommended.

**Literature**

**Textbooks**

**Useful references**


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

**Spinor Techniques in General Relativity (L24)**

*Non-Examinable (Graduate Level)*

Irena Borzym (12 Lectures) and Peter O’Donnell (12 Lectures)

Spinor structures and techniques are an essential part of modern mathematical physics. This course provides a gentle introduction to spinor methods which are illustrated with reference to a simple 2-spinor formalism in four dimensions. Apart from their role in the description of fermions, spinors also often provide useful geometric insights and consequent algebraic simplifications of some calculations which are cumbersome in terms of spacetime tensors.

The first half of the course will include an introduction to spinors illustrated by 2-spinors. Topics covered will include the conformal group on Minkowski space and a discussion of conformal compactifications, geometry of scri, other simple simple geometric applications of spinor techniques, zero rest mass field equations, Petrov classification, the Plucker embedding and a comparison with Euclidean spacetime. More specific references will be provided during the course and there will be worked examples and handouts provided during the lectures.
The second half of the course will include: Newman-Penrose (NP) spin coefficient formalism, NP field equations, NP quantities under Lorentz transformations, Geroch-Held-Penrose (GHP) formalism, modified GHP formalism, Goldberg-Sachs theorem, Lanczos potential theory, Introduction to twistors. There will be no problem sets.

Pre-requisites

The Part 3 general relativity course is a prerequisite. No prior knowledge of spinors will be assumed.

Literature

Introductory material.


Best Course Reference Text for Lectures 1 to 12.


Best Course Reference Text for Lectures 13 to 24.


Reading to complement course material.


Nonlinear Wave Equations (L8)

Non-Examinable (Graduate Level)

Joseph Keir

From fluid mechanics to general relativity, nonlinear wave equations are ubiquitous in physics, and this course will give an introduction to their analysis. Our approach will be based on energy methods, which take as their foundation the conservation of energy for linear wave equations, but turn out to be remarkably powerful and adaptable. The course will cover

- local theory for nonlinear wave equations
- global properties of solutions and the Klainerman-Sobolev inequality
- the null condition and global existence for nonlinear wave equations
- the weak null condition and a sketch of the proof of the stability of Minkowski space in general relativity
Pre-requisites

Some experience with partial differential equations as well as differential geometry will be useful, but most tools will be developed from scratch.

Literature

2. H. Ringstrom, *Non-linear wave equations*, available at
Introduction to Astrophysics courses

These courses provide a broad introduction to research in theoretical astrophysics; they are taken by students of both Part III Mathematics and Part III Astrophysics. The courses are mostly self-contained, building on knowledge that is common to undergraduate programmes in theoretical physics and applied mathematics. For specific pre-requisites please see the individual course descriptions.

Planetary System Dynamics (M24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of (well over 1000) extrasolar planets and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems.

Specific topics to be covered include:

1. Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation
2. Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler’s equation, perturbed orbits
3. Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift
4. Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits
5. Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide
6. Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces
7. Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange’s planetary equations, classification of arguments
9. Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap
Pre-requisites
This course is self-contained.

Literature

Additional support
Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

**Structure and Evolution of Stars (M24)**

A.N. Żytkow

Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a mathematical description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These basic equations have to be supplemented by a number of appropriately chosen equations describing the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed. Some familiarity with the principles of hydrodynamics, thermodynamics, quantum mechanics, atomic and nuclear physics will be assumed.

Approximate solutions of the equations will be shown; polytropic gas spheres, homology principles, the virial theorem will be presented. The evolution of a star will be discussed, starting from the main-sequence, following the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

The only way in which we may test stellar structure and evolution theory is through comparison of the theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

Pre-requisites
At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics although a detailed knowledge of all of these is not expected.

Preliminary Reading
Literature


Additional support

There will be four example sheets each of which will be discussed during an examples class. There will be a one-hour revision class in the Easter Term.

**Astrophysical Fluid Dynamics (M24)**

**Roman Rafikov**

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is ‘frozen in’ to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as their stability and their response to tidal forcing.

**Provisional synopsis**

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Stellar oscillations. Introduction to asteroseismology and astrophysical tides.
- Local dispersion relation. Internal waves and instabilities in stratified rotating astrophysical bodies.

**Pre-requisites**

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of fluid dynamics, thermodynamics and electromagnetism will be assumed.
Literature


Additional support

Four example sheets will be provided and four associated classes will be given by the lecturer. Extended notes supporting the lecture course are available from reference 6 in the list above. There will be a revision class in Easter Term.

**Optical and infrared astronomical telescopes and instruments (M16)**

Ian Parry

Astronomy is an observational science. Our understanding of the universe beyond the Earth comes mostly from interpreting the electromagnetic radiation we see coming from the sky. This course is about the equipment and techniques that we use to collect and measure the optical and near infra-red component of this radiation (approximately 0.3 to 5 microns in wavelength).

The material presented will give the student a thorough understanding of how telescopes and their instruments actually work. An important aim of the course is to quantify how well they work leading to an understanding of what defines the state-of-the-art and what its limitations are.

Specific topics will be selected from the following list:

1. Introduction: Effects of the Earth's atmosphere, transparency, seeing, refraction, dispersion, basic definition of magnitudes.
2. Positional astronomy and coordinate systems: Sidereal time, right ascension, declination, hour angle, aberration of starlight, spherical trigonometry, great circles, small circles, spherical triangles, cosine and sine rules, the analogue formula, tangent plane.
3. Optics: geometrical optics, lens-makers equation, principle planes, focal length, f-number, aberrations, paraxial approximation, ray-tracing, image planes, pupil planes, conjugate planes, simple lens design, achromatic lenses, the Petzval lens, methods of designing complex optical systems, étendue, physical optics, quantum optics, Fourier treatment of wave propagation.
4. Telescopes. Refractors, reflectors, parabolic reflectors, Ritchey-Chretien telescopes, 3-mirror anastigmats, the Schmidt telescope, ground-based telescopes, telescope mounts, space-based telescopes.
5. Detectors: photo-electrons, useful semiconductor types, readout architectures, image intensifiers, linearity, dynamic range, quantum efficiency, pixel-to-pixel variations, readout noise, dark-current, sensitivity to cosmic rays, defects, charge transfer, artefacts.
7. Imagers: eyepieces, the eye, magnification, collimators, cameras, detector matching, image scale, field of view, filters, magnitude systems. Fabry Perot interferometers, polarimetry.

8. Coronagraphs: Fourier modelling, speckles, occulters, lyot-stop, apodization,

9. Spectrographs: Dispersive spectrometers (long-slit, multi-slit, multi-fibre, echelle, integral field), disperser types, grating equation, spectro-polarimetry, Fourier-transform spectrometers.


12. Future projects: E-ELT, LSST, JWST, HDST.

Pre-requisites
This course is self-contained.

Literature
6. Optics : E.Hecht
7. Speckle Phenomenon in Optics : J.W.Goodman
8. Astronomical Optics : D.J.Shroeder
9. Astronomical Techniques : W.A.Hiltner

Additional support
Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extrasolar Planets: Atmospheres and Interiors (L24)  
Nikku Madhusudhan

The field of extrasolar planets (or ‘exoplanets’) is one of the most dynamic frontiers of modern astronomy. Exoplanets are planets orbiting stars beyond the solar system. Thousands of exoplanets are now known with a wide range of sizes, temperatures, and orbital parameters, covering all the categories of planets in the solar system (gas giants, ice giants, and rocky planets) and more. The field is now moving into a new era of Exoplanet Characterization, which involves understanding the atmospheres, interiors, and formation mechanisms of exoplanets, and ultimately finding potential biosignatures in the atmospheres of rocky exoplanets. These efforts are aided by both high-precision spectroscopic observations as well as detailed theoretical models of exoplanets.
The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and stratospheres, equilibrium/non-equilibrium chemistry, atmospheric dynamics, clouds/hazes, etc) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, as well as parametric models and retrieval techniques) (3) exoplanetary interiors (equations of state, mass-radius relations, and internal structures of giant planets, super-Earths, and rocky exoplanets), and (4) relating atmospheres and interiors to planet formation. Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres of all kinds. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed. The course will also include a discussion on detecting biosignatures in rocky exoplanets, the relevant theoretical constructs and expected observational prospects with future facilities.

Pre-requisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self-contained and basic concepts will be introduced as required.

Literature


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Dynamics of Astrophysical Discs (L16)

Henrik Latter

Discs of matter in orbital motion around a massive central body occur in numerous situations in astrophysics. For example, Saturn’s rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a (non-Newtonian) fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary star systems and in the centres of active galaxies; they reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with
the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of extrasolar planets.

Provisional synopsis:

- Occurrence of discs in various astronomical systems, basic physical and observational properties.
- Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.
- Viscous evolution of an accretion disc.
- Vertical disc structure, thin-disc approximations, thermal instability in cataclysmic variables.
- The shearing sheet, symmetries, shearing waves.
- Incompressible dynamics: hydrodynamic stability, vortices and dust dynamics in protoplanetary disks.
- Compressible dynamics: density waves, gravitational instability and ‘gravitoturbulence’ in planetary rings and protoplanetary discs.
- Satellite-disc interaction, impulse approximation, gap opening by embedded planets.
- Magnetorotational instability, ‘dead zones’ in protoplanetary discs.

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is helpful for the magnetorotational instability.

Literature


Additional support

Three examples sheets will be provided and three associated examples classes will be given.
Quantum Information Theory (L24)
Nilanjana Datta

Quantum Information Theory (QIT) is an exciting, young field which lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the mathematical theory of acquisition, storage, transmission and processing of information. QIT is the study of how these tasks can be accomplished, using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical analogues. These new features can be exploited, not only to improve the performance of certain information-processing tasks, but also to accomplish tasks which are impossible or intractable in the classical realm.

This is an introductory course on QIT, which should serve to pave the way for more advanced topics in this field. The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of QIT. Topics in this part of the course will include a brief discussion of data compression, transmission of data through noisy channels, Shannon’s theorems, entropy and channel capacity.

The quantum part of the course will commence with a study of open systems and a discussion of how they necessitate a generalization of the basic postulates of quantum mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs, the Kraus Representation Theorem and the Choi-Jamilkowski isomorphism. Entanglement and some applications elucidating its usefulness as a resource in QIT will be discussed. This will be followed by a study of the von Neumann entropy, its properties and its interpretation as the data compression limit of a quantum information source. Schumacher’s theorem on quantum data compression will be discussed in detail. The definitions of ensemble average fidelity and entanglement fidelity will be introduced in this context. Definitions and properties of the quantum conditional entropy, quantum mutual information, the quantum relative entropy and coherent information will be discussed. Various examples of quantum channels will be given and the different capacities of a quantum channel will be discussed. The Holevo bound on the accessible information and the Holevo-Schumacher-Westmoreland (HSW) Theorem will also be covered.

Pre-requisite Mathematics

Knowledge of basic quantum mechanics will be assumed. However, an additional lecture can be arranged for students who do not have the necessary background in quantum mechanics. Elementary knowledge of Probability Theory, Vector Spaces and Linear Algebra will be useful.

Literature

The following books and lecture notes provide interesting and relevant reading material.
3. J.Preskill, Chapter 5 of his lecture notes: Lecture notes on Quantum Information Theory, https://www.theory.caltech.edu
Philosophy of Physics

The courses in Philosophy of Physics are open to all students doing Part III, but are formally listed as graduate courses. This means there is no exam at the end of May for any such course; but a Part III student can get credit for them by doing their submitted Part III essay in association with one of the courses. More generally, the Philosophy of Physics courses are intended as a refreshing and reflective companion to the other Part III courses, especially the courses in theoretical physics.

Philosophical Aspects of Quantum Field Theory (M8)

Non-Examinable (Part III Level)

J. Butterfield

Quantum field theory has for many decades been the framework for several basic and outstandingly successful physical theories. Nowadays, it is being addressed by philosophy of physics (which has traditionally concentrated on conceptual questions raised by non-relativistic quantum mechanics and relativity). This course will introduce this literature. The content will be moulded by students’ interests. But I hope: (i) to emphasize quantization theory, and algebraic methods; (ii) to mostly use the books by Folland, and by de Faria and de Melo; and (iii) to lead up to the Unruh effect.

I also expect, in the first half of the course, to review: (a) the mathematical structure of quantum theories in general, at the level of the books by Hannabuss, Jordan and Prugovecki; (b) some foundational issues, using the books by Araki, Clifton and Landsman (which is Open Access); (c) ideas of operator algebras, using the books by Emch, Haag and Ruetsche.

Pre-requisites

There are no formal prerequisites. Previous familiarity with quantum field theory, such as provided by the Part III courses, will be helpful.

Preliminary Reading

This list of reading gives an overview of the course’s topics, and is approximately in order of increasing difficulty.


Literature

The main resource will be the books by Folland, and by de Faria and de Melo. For mathematical background, we will draw on the books by Hannabuss, Jordan and Prugovecki. For foundational issues, we will draw on the books by Araki, Clifton and Landsman (the last being freely downloadable, and an invaluable resource for the whole course). For operator algebras, we will also use the books by Emch, Haag and Ruetsche. An overall reference is Ticciati (1999); a recent advanced monograph is Rejzner (2016).


Additional support

One or two Part III essays—one of them probably about the Unruh effect—will be offered in conjunction with this course.

**Hamiltonian General Relativity (L8)**

*Non-Examinable (Part III Level)*

**J. B. Pitts**

How does one set up a Hamiltonian formulation for GR, Maxwell or Yang-Mills, given that the Legendre transformation does not work? Does Hamiltonian General Relativity really lack change? Are observables nonlocal and constants of the motion? Such claims are part of the “problem of time” that has afflicted Hamiltonian General Relativity (and hence canonical quantum gravity!) since the 1950s. These lectures are a foundationally-oriented introduction to Rosenfeld-Dirac-Bergmann constrained Hamiltonian dynamics with special attention to real examples and the relation to the Lagrangian and 4-dimensional geometric formalisms.
Pre-requisites

Familiarity with mechanics, electromagnetism, and perhaps a bit of General Relativity is helpful but is not required.

Literature

15. R. Wald, General Relativity, appendix E.

Additional support

A Part III essay will be offered in conjunction with this course.
In an increasingly data driven world, there is an essential need for efficient acquisition techniques, the correct representation of signals and the ability to extract meaningful information from signals. This is an introductory course to the mathematics behind such techniques. This course will cover the following topics:

- The representation and approximation of signals, in particular, linear and nonlinear approximation in Fourier and wavelet bases.
- Efficient data acquisition by exploiting the inherent sparse structure of signals, in particular, an introduction to compressed sensing.
- Topics in inverse problems, in particular, the use of total variation regularization to exploit the geometric structures of the underlying signals.

Pre-requisites

This course assumes knowledge in analysis and linear algebra. Additional knowledge in partial differential equations, functional analysis, variational calculus is beneficial, but not mandatory.

Literature


Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Distribution Theory and Applications (M16)

A.C.L. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the use of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will look at the Sobolev spaces $H^s(\mathbb{R}^n)$ and $H^s_{\text{loc}}(X)$ and their description in terms of the Fourier transform of tempered distributions. Time permitting, the material that follows will address questions such as

- What does a generic distribution look like?
• Why are solutions to Laplace’s equation always infinitely differentiable?
• Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. In the final part of the course we will study Hörmander’s oscillatory integrals.

Pre-requisites
Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Analysis/Methods). No knowledge of functional analysis is assumed.

Preliminary Reading

Literature

Additional support
Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be made available.

Topics in Convex Optimisation (M16)

Hamza Fawzi

Mathematical optimisation problems arise in many areas of science and engineering, including statistics, machine learning, robotics, signal/image processing, and others. This course will cover some techniques known as *convex relaxations*, to deal with optimisation problems involving polynomials, which are in general intractable. The emphasis of the course will be on semidefinite programming which is a far-reaching generalization of linear programming. A tentative list of topics that we will cover include:

• From linear programming to conic programming. Duality theory.
• Semidefinite optimisation and convex relaxations. Sums-of-squares and moment problems.
• Applications: binary quadratic optimisation and rounding methods (e.g., Goemans-Williamson rounding), stability of dynamical systems, matrix completion/low-rank matrix recovery, etc.

Pre-requisites
This course assumes basic knowledge in linear algebra and analysis. Some knowledge of convex analysis will be useful.
Literature


Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

**Numerical Solution of Differential Equations (L24)**

**Arieh Iserles**

The course will address modern algorithms for the solution of ordinary and partial differential equations, inclusive of finite difference and finite element methods, with an emphasis on broad mathematical principles underlying their construction and analysis.

**Pre-requisites**

Although prior knowledge of some numerical analysis and of abstract function spaces is advantageous, it will not be taken for granted. Reasonable understanding of basic concepts of analysis (complex analysis and analytic functions, basic existence and uniqueness theorems for ODEs and PDEs, elementary facts about PDEs) and of linear algebra is a prerequisite.

**Literature**


**Additional support**

An extensive printed handout, covering the entire material of the course, will be provided in the first week. There will be weekly examples’ classes, starting from the third week, as well as a revision supervision in the Easter Term.

**Inverse Problems (L24)**

**Matthias Ehrhardt & Lukas Lang**

Solving an inverse problem is the task of computing an unknown quantity from observed measurements. Inverse problems are among the most important problems in a variety of subjects such as physics, biology, medicine, engineering, and finance; including tomography (e.g., computed tomography (CT)), machine learning, computer vision, and image processing. Computing a solution to an inverse problem is not straightforward for three basic reasons: either it may not exist, may not be unique, or small errors in the measurements get heavily amplified which renders the solution useless. In this course we address mathematical aspects of inverse problems that are needed to find stable and meaningful solutions—from founding concepts to modern numerical algorithms.
Pre-requisites

This course assumes basic knowledge in linear algebra and analysis (e.g. linear analysis or analysis of functions).

Literature


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Boundary Value Problems for Linear PDEs (L16)

Iasonas Hitzazis

Recent developments in the area of the so-called *integrable nonlinear* Partial Differential Equations (PDEs) have led to the emergence of a new method for solving boundary value problems, which is usually referred to as the *Unified Transform* (UT).

The UT will be implemented to:

(a) Linear evolution PDEs in one spatial variable formulated either on the half-line or on a finite interval. Examples include the heat equation and the Stokes equation (linearised version of the KdV).

(b) Linear elliptic PDEs in two spatial variables formulated in the interior of a convex polygon. Examples include the Laplace, the modified Helmholtz, and the Helmholtz equations.

For the above problems, in addition to presenting integral representations of the solution, simple numerical techniques for the effective computation of the solution will also be introduced.

Pre-requisites

The course only requires some elementary knowledge of complex analysis.

Literature


Additional support

Three examples sheets will be provided and three associated examples classes will be given.
Variational Methods and PDE (E8)

Non-Examinable (Graduate Level)

Dr P. Markowich

The course deals with variational (minimization) problems in integral form and their associated Euler-Lagrange partial differential equations. Topics are lower semicontinuity, coercivity convexity, polyconvexity of functionals, obstacle problems, Hamiltonian-Lagrangian duality, gradient flows and the mountain pass theorem.

Pre-requisites

Partial differential equations, functional analysis and Sobolev space

Literature

2. *Graduate Studies in Mathematics Volume: 19* 2010; 749 pp
The four courses in the Michaelmas Term are intended to provide a broad educational background for any student preparing to start a PhD in fluid dynamics. The courses in the Lent Term are more specialized and in some cases (see the course descriptions) build on the Michaelmas Term material.

**Desirable previous knowledge**

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice, familiarity with the continuum assumption, the material derivative, the stress tensor and the Navier-Stokes equation will be assumed, as will basic ideas concerning incompressible, inviscid fluid mechanics (e.g. Bernoulli’s Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable. Previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is desirable for some courses. No previous knowledge of solid mechanics, Earth Sciences, or biology is required.

In summary, knowledge of Chapters 1-8 of ‘Elementary Fluid Dynamics’ (D.J. Acheson, Oxford), plus Chapter 3 of ‘Waves in Fluids’ (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace’s equation, Poisson’s equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses

<table>
<thead>
<tr>
<th>Year</th>
<th>Courses</th>
</tr>
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<tbody>
<tr>
<td>First</td>
<td>Differential Equations, Dynamics and Relativity, Vector Calculus, Vectors &amp; Matrices.</td>
</tr>
<tr>
<td>Second</td>
<td>Methods, Complex Methods, Fluid Dynamics.</td>
</tr>
<tr>
<td>Third</td>
<td>Fluid Dynamics, Waves, Asymptotic Methods.</td>
</tr>
</tbody>
</table>

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses on WWW with URL:

[http://www.maths.cam.ac.uk/undergrad/schedules/](http://www.maths.cam.ac.uk/undergrad/schedules/)

**Hydrodynamic Stability (M24)**

**Colm-cille Caulfield**

Developing an understanding by which “small” perturbations grow, saturate and modify fluid flows is central to addressing many challenges of interest in fluid mechanics. Furthermore, many applied mathematical tools of much broader relevance have been developed to solve hydrodynamic stability problems, and hydrodynamic stability theory remains an exceptionally active area of research, with several exciting new developments being reported over the last few years.

In this course, an overview of some of these recent developments will be presented. After an introduction to the general concepts of flow instability, presenting a range of examples, the major content of this course will be focussed on the broad class of flow instabilities where velocity “shear” and fluid inertia play key dynamical roles. Such flows, typically characterised by sufficiently “high” Reynolds number $Ud/\nu$, where $U$ and $d$ are characteristic velocity and length scales of the flow, and $\nu$ is the kinematic viscosity of the fluid, are central to modelling flows in the environment and industry. They typically demonstrate the key role played by the redistribution of vorticity within the flow, and such vortical flow instabilities often trigger the complex, yet hugely important process of “transition to turbulence”.

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A hierarchy of mathematical approaches will be discussed to address a range of “stability” problems, from more traditional concepts of “linear” infinitesimal normal mode perturbation energy growth on laminar parallel shear flows to transient, inherently nonlinear perturbation growth of general measures of perturbation magnitude over finite time horizons where flow geometry and/or fluid properties play a dominant role. The course will also discuss in detail physical interpretations of the various flow instabilities considered, as well as the industrial and environmental application of the results of the presented mathematical analyses.

Pre-requisites

Undergraduate fluid mechanics, linear algebra, complex analysis and asymptotic methods.

Literature

1. F. Charru *Hydrodynamic Instabilities* CUP 2011.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

**Slow Viscous Flow (M24)**

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth’s mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.
Pre-requisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Preliminary Reading


Literature


Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Fluid Dynamics of the Environment (M24)

S.B. Dalziel and N.M. Vriend

Understanding the environment and predicting the impact of human activity on it are critical challenges in our time. Whether we are concerned about climate change, pollution or thermal comfort within our buildings, the fluid dynamics of bodies of water (rivers, lakes and oceans) and the atmospheres play a vital role. This course introduces the basic fluid dynamics necessary to build mathematical models of the environment in which we live, and focuses on problems which occur over sufficiently small time and length scales to be largely unaffected by the earth’s rotation.

The course begins by considering the fluid flow in the presence of (typically small) density variations. If the fluid is stably stratified, ‘internal gravity waves can occur since the stratification provides a restoring force when fluid parcels are displaced vertically. The course highlights some of the rich and surprising dynamics of these waves. For example, internal gravity waves radiate energy vertically as well as horizontally, and their interaction with boundaries can focus this energy and cause mixing far from where the energy was input.

Density variations within fluids can also drive the flow and the course will consider two important and related classes where the flow is either long and shallow or tall and thin. Both classes allow substantial simplification of the governing equations by integrating them over the smaller dimension. In the first, when there are lateral gradients in fluid density interacting with horizontal or sloping boundaries, turbulent ‘density or gravity currents can develop. For the second, a relatively localised source can drive the rise of a turbulent ‘plume’ of buoyant fluid. Volcanic eruption clouds and accidental releases of pollution are just two examples of such plumes.

The buoyancy driving these flows may be due to differences in temperature or composition (e.g. salt or water vapour concentration), or due to the presence of a second phase such as particles or bubbles. Examples of particle-laden flows include snow avalanches, turbidity currents and pyroclastic flows. Particle-fluid and particle-particle interactions introduce a new range of interesting features. Particle suspension and deposition are important in a broad range of phenomena such as dune building and sand transport.
Pre-requisites

Undergraduate fluid dynamics is desirable.

Literature

Reading to complement course material


Additional support

In addition to the lectures, four examples sheets will be provided and four associated examples classes will run in parallel to the course. There will be a revision class in the Easter Term.

Perturbation Methods (M16)

L.J. Ayton & S.J. Cowley

This course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called ‘exponential asymptotics’), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- **Methods for Approximating Integrals.** This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the ‘divide & conquer’ strategy, Laplace’s method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to ‘asymptotics beyond all orders’ in which exponentially small corrections are extracted from the tails of asymptotic series. The advantage of uniformly valid expansions for comparison with experiment and numerical solutions will be covered. [7]

- **Matched Asymptotic Expansions.** This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. Further examples will be given of asymptotics beyond all orders. This section will include a brief introduction to the summation of [divergent] series, e.g. covering Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks’ transformations, Richardson extrapolation, and Domb-Sykes plots. [6]

- **Multiple Scales.** This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the ‘WKB[JLG]’ method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of ‘averaging’ and ‘time scale distortion’ in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium). [3]
Pre-requisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward differential equations and partial differential equations and evaluate simple integrals.

Literature

Relevant Textbooks

1. Bender, C.M. & Orszag, S., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to ‘Stokes’ lines as ‘anti-Stokes’ lines, and vice versa. The course will use Stokes’ convention.*


Reading to Complement Course Material


Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.

Fluid dynamics of the solid Earth (L24)

Jerome A. Neufeld & M. Grae Worster

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth’s core is formed by the solidification of a mixture of molten iron and various lighter elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth’s magnetic field. At very much longer time scales, radiogenic heating of the solid mantle drives solid-state convection resulting in plume-like features possibly responsible for features such as the Hawaiian sea mounts. Nearer the surface, convection drives the motion of brittle plates which are responsible for the Earth’s topography as can be felt and imaged through the seismic record. Upwelling mantle material also drives partial melting of mantle rocks resulting in compaction, and ultimately in the propagation of viscous melt through the elastic crust. On the Earth’s surface, and at very much faster rates, the same physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth’s cryosphere, from the solidification of sea ice to the flow of glacial ice.

This course will use the wealth of observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical
topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials.

**Pre-requisites**

A basic understanding of viscous fluid dynamics. Mathematical methods, particularly the solution of ordinary and partial differential equations.

**Literature**


**Additional support**

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

**Fluid Dynamics of Climate (L24)**

**P.H. Haynes  J.R. Taylor**

Understanding the Earth’s climate and predicting its future evolution is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the climate system, helping to make the planet hospitable for life. The dynamical complexity of this fluid motion and the wide range of space and time scales involved is one of the most difficult aspects of climate prediction.

This course, focusing on the large-scale behaviour of stratified and rotating flows, provides an introduction to the fluid dynamics necessary to build mathematical models of the climate system. The course begins by considering flows which evolve on a timescale which is long compared with a day, where the Earth's rotation plays an important role. The rotation is felt through the Coriolis force (a fictitious force arising from use of a frame of reference rotating with the Earth) which causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics, particularly in combination with stable density stratification. Canonical models are introduced and studied to illustrate phenomena such as adjustment to a state of geostrophic balance, where Coriolis force balances pressure gradient, new wave modes that can communicate dynamical information on both regional and global scales, and new hydrodynamic instabilities that lead to atmospheric weather systems and ocean eddies.

The course then moves on to apply these basic ideas to important aspects of the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the structure and hence the effects of eddies and weather systems, the dynamics of ocean gyres and boundary currents like the Gulf Stream, the dynamics of the meridional (north/south) circulation in the ocean and atmosphere and the associated transport of heat and of chemical and biological tracers and special dynamics of tropical regions which give rise to phenomena such as El Nino.

**Desirable Previous Knowledge**

Undergraduate fluid dynamics
Reading to complement course material


Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Direct and Inverse Scattering of Waves (L16)

Orsola Rath Spivack

The study of wave scattering is concerned with how the propagation of waves is affected by objects, and has a variety of applications in many fields, from environmental science to seismology, medicine, telecommunications, materials science, military applications, and many others. If we know the nature of the objects and we want to find how an incident wave is scattered, we call this a ‘direct scattering problem’ and practical applications will include for example underwater sound propagation, light transmission through the atmosphere, or the effect of noise in built-up areas. If we measure and know the scattered field produced by an incident wave, but we do not know the nature of the objects that have scattered it, we call this an ‘inverse scattering problem’ and applications will include for example non-destructive testing of materials, remote sensing with radar or lidar, or medical imaging.

This course will provide the basic theory of wave propagation and scattering and an overview of the main mathematical methods and approximations, with particular emphasis on inhomogeneous and random media, and on the regularisation of inverse scattering problems. Only time-harmonic waves will be normally considered.

Topics covered will include:
1. Boundary value problems and the integral form of the wave equation.
2. The parabolic equation and Born and Rytov approximations for the scattering problem.
3. Scattering by randomly rough surfaces and propagation in inhomogeneous media.
5. Regularisation methods and methods for solving some inverse scattering problems.
6. Time reversal and focusing in inhomogeneous media.

Pre-requisites

This course assumes basic knowledge of ODEs and PDEs, and of Fourier transforms. Some familiarity with linear algebra and with basic concepts in functional analysis is helpful, though by no means necessary.

Students doing this course might find it helpful, though by no means necessary, to attend also the following Part III courses: “Topics in Convex Optimisation”, and “Inverse Problems”.

Preliminary Reading


**Literature**


**Additional support**

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

**Theoretical Physics of Soft Condensed Matter (L16)**

Mike Cates

Soft Condensed Matter refers to liquid crystals, emulsions, molten polymers and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, generally with noise so that the equations of motion are stochastic PDEs. The form of these equations is helpfully constrained by the requirement that the Boltzmann distribution is regained in the steady state (when this indeed holds, i.e. for systems in contact with a heat bath but not subject to forcing). Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a ‘smectic liquid crystal’ (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field (and several steps short of the renormalization group, whose application to classical field theories is discussed in other courses but not this one). Important models of soft matter include diffusive $\phi^4$ field theory (‘Model B’), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates ‘Model H’, a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth. We will also explore the continuum theory of nematic liquid crystals, which spontaneously break rotational but not translational symmetry, focusing on topological defects and their associated mathematical structure such as homotopy classes.

Finally, the course will cover some recent extensions of the same general approach to systems whose microscopic dynamics does not have time-reversal symmetry, such as self-propelled colloidal swimmers. These systems do not have a Boltzmann distribution in steady state; without that constraint, new field theories arise that are the subject of ongoing research.
Pre-requisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the following Michaelmas Term courses although none are prerequisites: Statistical Field Theory; Biological Physics and Complex Fluids; Slow Viscous Flow; Quantum Field Theory.

Preliminary Reading

1. D. Tong *Lectures on Statistical Physics*
   
   [http://www.damtp.cam.ac.uk/user/tong/statphys.html](http://www.damtp.cam.ac.uk/user/tong/statphys.html)

   Before embarking on this course you do need to understand the equation \( F = -k_B T \ln Z \) and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Familiarity with the Landau theory of phase transitions is highly desirable. We will not need much abstract thermodynamics (e.g. Maxwell relations) but you do need to know the zeroth, first and second laws. These lecture notes are an excellent resource for revising and reviewing the key material.

   

   This set of lecture notes addresses only one part of the course (emulsions); it goes into more depth in that area than we will, but with more words and significantly less mathematics. It takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics when needed. Despite all this, these notes would make useful preliminary reading and gives an idea of the types of problem we will address.

Literature

I am not aware of any books that treat this material at the right level. But it may be worth looking at:

1. P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics.* Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, topological defects all feature in the description of condensed matter systems at \( \hbar = 0 \). More for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will also be a two-hour revision class in the Easter Term.

**Demonstrations in Fluid Mechanics. (L8)**

*Non-Examinable (Part III Level)*

Prof. S.B. Dalziel, Dr. J.A. Neufeld

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments
play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive ‘feeling’ for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- porous media and carbon sequestration;
- fluid flow and elastic deformation;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

**Pre-requisites**

Undergraduate Fluid Dynamics.

**Literature**

