You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
SECTION I

1A Vectors and Matrices

Let \( z \in \mathbb{C} \) be a solution of

\[ z^2 + bz + 1 = 0, \]

where \( b \in \mathbb{R} \) and \( |b| \leq 2 \). For which values of \( b \) do the following hold?

(i) \( |e^z| < 1 \).
(ii) \( |e^{iz}| = 1 \).
(iii) \( \text{Im}(\cosh z) = 0 \).

2C Vectors and Matrices

Write down the general form of a \( 2 \times 2 \) rotation matrix. Let \( R \) be a real \( 2 \times 2 \) matrix with positive determinant such that \( |Rx| = |x| \) for all \( x \in \mathbb{R}^2 \). Show that \( R \) is a rotation matrix.

Let

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Show that any real \( 2 \times 2 \) matrix \( A \) which satisfies \( AJ = JA \) can be written as \( A = \lambda R \), where \( \lambda \) is a real number and \( R \) is a rotation matrix.

3D Analysis I

What does it mean to say that a sequence of real numbers \( (x_n) \) converges to \( x \)? Suppose that \( (x_n) \) converges to \( x \). Show that the sequence \( (y_n) \) given by

\[ y_n = \frac{1}{n} \sum_{i=1}^{n} x_i \]

also converges to \( x \).

4F Analysis I

Let \( a_n \) be the number of pairs of integers \( (x, y) \in \mathbb{Z}^2 \) such that \( x^2 + y^2 \leq n^2 \). What is the radius of convergence of the series \( \sum_{n=0}^{\infty} a_n z^n \)? [You may use the comparison test, provided you state it clearly.]

Part IA, Paper 1
SECTION II

5A Vectors and Matrices

(a) Use suffix notation to prove that
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \].

(b) Show that the equation of the plane through three non-collinear points with position vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is
\[ \mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \]
where \( \mathbf{r} \) is the position vector of a point in this plane.

Find a unit vector normal to the plane in the case \( \mathbf{a} = (2, 0, 1), \mathbf{b} = (0, 4, 0) \) and \( \mathbf{c} = (1, -1, 2) \).

(c) Let \( \mathbf{r} \) be the position vector of a point in a given plane. The plane is a distance \( d \) from the origin and has unit normal vector \( \mathbf{n} \), where \( \mathbf{n} \cdot \mathbf{r} \geq 0 \). Write down the equation of this plane.

This plane intersects the sphere with centre at \( \mathbf{p} \) and radius \( q \) in a circle with centre at \( \mathbf{m} \) and radius \( \rho \). Show that
\[ \mathbf{m} - \mathbf{p} = \gamma \mathbf{n} \].

Find \( \gamma \) in terms of \( q \) and \( \rho \). Hence find \( \rho \) in terms of \( \mathbf{n}, d, \mathbf{p} \) and \( q \).
6B Vectors and Matrices

The \( n \times n \) real symmetric matrix \( M \) has eigenvectors of unit length \( e_1, e_2, \ldots, e_n \), with corresponding eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \), where \( \lambda_1 > \lambda_2 > \cdots > \lambda_n \). Prove that the eigenvalues are real and that \( e_a \cdot e_b = \delta_{ab} \).

Let \( x \) be any (real) unit vector. Show that
\[
x^T M x \leq \lambda_1.
\]

What can be said about \( x \) if \( x^T M x = \lambda_1 \)?

Let \( S \) be the set of all (real) unit vectors of the form
\[x = (0, x_2, \ldots, x_n).
\]

Show that \( \alpha_1 e_1 + \alpha_2 e_2 \in S \) for some \( \alpha_1, \alpha_2 \in \mathbb{R} \). Deduce that
\[
\max_{x \in S} x^T M x \geq \lambda_2.
\]

The \((n - 1) \times (n - 1)\) matrix \( A \) is obtained by removing the first row and the first column of \( M \). Let \( \mu \) be the greatest eigenvalue of \( A \). Show that
\[
\lambda_1 \geq \mu \geq \lambda_2.
\]

7B Vectors and Matrices

What does it mean to say that a matrix can be diagonalised? Given that the \( n \times n \) real matrix \( M \) has \( n \) eigenvectors satisfying \( e_a \cdot e_b = \delta_{ab} \), explain how to obtain the diagonal form \( \Lambda \) of \( M \). Prove that \( \Lambda \) is indeed diagonal. Obtain, with proof, an expression for the trace of \( M \) in terms of its eigenvalues.

The elements of \( M \) are given by
\[
M_{ij} = \begin{cases} 0 & \text{for } i = j, \\ 1 & \text{for } i \neq j. \end{cases}
\]

Determine the elements of \( M^2 \) and hence show that, if \( \lambda \) is an eigenvalue of \( M \), then
\[
\lambda^2 = (n - 1) + (n - 2) \lambda.
\]

Assuming that \( M \) can be diagonalised, give its diagonal form.
8C Vectors and Matrices

(a) Show that the equations

\[ \begin{align*}
1 + s + t &= a \\
1 - s + t &= b \\
1 - 2t &= c
\end{align*} \]

determine \( s \) and \( t \) uniquely if and only if \( a + b + c = 3 \).

Write the following system of equations

\[ \begin{align*}
5x + 2y - z &= 1 + s + t \\
2x + 5y - z &= 1 - s + t \\
-x - y + 8z &= 1 - 2t
\end{align*} \]

in matrix form \( Ax = b \). Use Gaussian elimination to solve the system for \( x, y, \) and \( z \).

State a relationship between the rank and the kernel of a matrix. What is the rank and what is the kernel of \( A \)?

For which values of \( x, y, \) and \( z \) is it possible to solve the above system for \( s \) and \( t \)?

(b) Define a unitary \( n \times n \) matrix. Let \( A \) be a real symmetric \( n \times n \) matrix, and let \( I \) be the \( n \times n \) identity matrix. Show that \( |(A + iI)x|^2 = |Ax|^2 + |x|^2 \) for arbitrary \( x \in \mathbb{C}^n \), where \( |x|^2 = \sum_{j=1}^n |x_j|^2 \). Find a similar expression for \( |(A - iI)x|^2 \). Prove that \( (A - iI)(A + iI)^{-1} \) is well-defined and is a unitary matrix.

9E Analysis I

State the Bolzano–Weierstrass theorem. Use it to show that a continuous function \( f : [a, b] \to \mathbb{R} \) attains a global maximum; that is, there is a real number \( c \in [a, b] \) such that \( f(c) \geq f(x) \) for all \( x \in [a, b] \).

A function \( f \) is said to attain a local maximum at \( c \in \mathbb{R} \) if there is some \( \varepsilon > 0 \) such that \( f(c) \geq f(x) \) whenever \( |x - c| < \varepsilon \). Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is twice differentiable, and that \( f''(x) < 0 \) for all \( x \in \mathbb{R} \). Show that there is at most one \( c \in \mathbb{R} \) at which \( f \) attains a local maximum.

If there is a constant \( K < 0 \) such that \( f''(x) < K \) for all \( x \in \mathbb{R} \), show that \( f \) attains a global maximum. [Hint: if \( g'(x) < 0 \) for all \( x \in \mathbb{R} \), then \( g \) is decreasing.]

Must \( f : \mathbb{R} \to \mathbb{R} \) attain a global maximum if we merely require \( f''(x) < 0 \) for all \( x \in \mathbb{R} \)? Justify your answer.
10E Analysis I

Let \( f : \mathbb{R} \to \mathbb{R} \). We say that \( x \in \mathbb{R} \) is a real root of \( f \) if \( f(x) = 0 \). Show that if \( f \) is differentiable and has \( k \) distinct real roots, then \( f' \) has at least \( k - 1 \) real roots. [Rolle’s theorem may be used, provided you state it clearly.]

Let \( p(x) = \sum_{i=1}^{n} a_i x^{d_i} \) be a polynomial in \( x \), where all \( a_i \neq 0 \) and \( d_{i+1} > d_i \). (In other words, the \( a_i \) are the nonzero coefficients of the polynomial, arranged in order of increasing power of \( x \).) The number of sign changes in the coefficients of \( p \) is the number of \( i \) for which \( a_i a_{i+1} < 0 \). For example, the polynomial \( x^5 - x^3 - x^2 + 1 \) has 2 sign changes. Show by induction on \( n \) that the number of positive real roots of \( p \) is less than or equal to the number of sign changes in its coefficients.

11D Analysis I

If \( (x_n) \) and \( (y_n) \) are sequences converging to \( x \) and \( y \) respectively, show that the sequence \((x_n + y_n)\) converges to \( x + y \).

If \( x_n \neq 0 \) for all \( n \) and \( x \neq 0 \), show that the sequence \( \left( \frac{1}{x_n} \right) \) converges to \( \frac{1}{x} \).

(a) Find \( \lim_{n \to \infty} \left( \sqrt{n^2 + n} - n \right) \).

(b) Determine whether \( \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} \) converges.

Justify your answers.

12F Analysis I

Let \( f : [0, 1] \to \mathbb{R} \) satisfy \(|f(x) - f(y)| \leq |x - y| \) for all \( x, y \in [0, 1] \).

Show that \( f \) is continuous and that for all \( \varepsilon > 0 \), there exists a piecewise constant function \( g \) such that

\[
\sup_{x \in [0,1]} |f(x) - g(x)| \leq \varepsilon.
\]

For all integers \( n \geq 1 \), let \( u_n = \int_{0}^{1} f(t) \cos(nt) dt \). Show that the sequence \( (u_n) \) converges to 0.

**END OF PAPER**