2.3 Continued Fractions

*This project requires an understanding of the Part IA course Numbers and Sets.*

1 Introduction

Every rational number \( x \) has a (finite) continued fraction expansion

\[
x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_N}}},
\]

where \( a_0 \) is an integer, and \( a_1, \ldots, a_N \) are positive integers with \( a_N > 1 \). It is customary to use the short-hand \( x = [a_0, a_1, \ldots, a_N] \). The continued fraction algorithm starts with \( x_0 = x \) and computes the sequence of partial quotients \( a_n \) by the transformations

\[
a_n = \lfloor x_n \rfloor
\]

\[
x_{n+1} = \frac{1}{x_n - a_n}
\]

where as usual \( \lfloor x \rfloor \) denotes the greatest integer \( \leq x \). The algorithm terminates when \( x_n = a_n \).

The convergents \( p_n/q_n \) are defined for \( n \geq 0 \) by

\[
p_n = a_np_{n-1} + p_{n-2}
\]

\[
q_n = a_nq_{n-1} + q_{n-2}
\]

with initial conditions \( p_{-2} = 0, p_{-1} = 1 \) and \( q_{-2} = 1, q_{-1} = 0 \). It can be shown by induction that \( p_n/q_n = [a_0, a_1, \ldots, a_n] \).

2 Continued fractions of rational numbers

**Question 1** Write a program to compute the continued fraction of a rational number \( u/v \). Your program should work with integers as far as possible, so that there is no risk of rounding errors. In some programming languages it is best to use an integer type, but there is no particular advantage in doing this if you are using MATLAB.

Compute the partial quotients and convergents for

\[
\frac{73}{13}, \quad \frac{3357}{2501}, \quad \frac{86497}{22592} \quad \text{and} \quad \frac{8920715}{6591846}.
\]

Estimate the complexity of your algorithm - that is, estimate the number of arithmetic operations needed in the worst case, as a function of \( u \) and \( v \).

We consider the \( 2 \times 2 \) matrices \( M_n = \begin{pmatrix} p_n & (-1)^{n-1}p_{n-1} \\ q_n & (-1)^{n-1}q_{n-1} \end{pmatrix} \).
**Question 2**  
Find a matrix identity relating $M_n$ and $M_{n-1}$. Use this identity to compute $p_n q_{n-1} - p_{n-1} q_n$.

Explain how your program from Question 1 can be modified to compute the highest common factor $d$ of integers $u$ and $v$, together with integers $r$ and $s$ such that $ru + sv = d$. Makes these changes to your program, and run it on the following examples:

$u = 513613 \quad u = 5497085 \quad u = 38541399$

$v = 90297659 \quad v = 81210519 \quad v = 51345565$.

**3 Solving linear congruences**

Euclid’s algorithm is an efficient way to compute the highest common factor of two integers. Importantly, it also provides a means to find integers $r$ and $s$ such that $ru + sv = \text{hcf}(u,v)$.

**Question 3**  
If you did not do so already, explain how your answers to Questions 1 and 2 are related to Euclid’s algorithm.

**Question 4**  
Describe clearly how Euclid’s algorithm can be used to find all the solutions in the unknown $x$ to the linear congruence $ax \equiv b \pmod{m}$.

**Question 5**  
Implement a routine to solve the linear congruence $ax \equiv b \pmod{m}$. Find all solutions to each of the following congruences. If none exist, state why not.

$294749x \equiv 177188 \pmod{408537}$

$436031x \equiv 363437 \pmod{570757}$

$403795x \equiv 411162 \pmod{787713}$

**4 Generators for the group $\text{SL}_2(\mathbb{Z})$**

Let $\text{SL}_2(\mathbb{Z})$ be the group of $2 \times 2$ integer matrices with determinant 1, under matrix multiplication. In the next question you will show that $\text{SL}_2(\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$  

**Question 6**  
Show that if $A_1$ and $A_2$ are matrices in $\text{SL}_2(\mathbb{Z})$ with the same first column, then $A_1 = A_2 T^n$ for some integer $n$.

Explain how your program from Question 1 can be modified to write an arbitrary element of $\text{SL}_2(\mathbb{Z})$ as a word in $S$, $T$ and $T^{-1}$, for example

$$\begin{pmatrix} 48 & 59 \\ 13 & 16 \end{pmatrix} = T^3 S T^{-1} S T^2 S T^{-3}.$$  

Make these changes to your program, and run it on the following examples:

$$\begin{pmatrix} 1484 & 1899 \\ 5253 & 6722 \end{pmatrix}, \quad \begin{pmatrix} 15415 & 33058 \\ 27253 & 58445 \end{pmatrix}, \quad \begin{pmatrix} 120883 & 223890 \\ 385864 & 714667 \end{pmatrix}.$$  

You should check your answers by multiplying out.
5 Continued fractions of real numbers

The partial quotients $a_n$ and convergents $p_n/q_n$ of an irrational number $x$ are computed as described in the introduction, except that the algorithm will no longer terminate. It can be shown that $p_n/q_n \to x$ as $n \to \infty$, so it makes sense to write $x = [a_0, a_1, a_2, \ldots]$.

Your computer almost certainly represents real numbers internally as a special sort of rational (usually with a denominator which is a power of 2), which thus represents all the real numbers in a certain interval. Hence most real numbers cannot be stored exactly, but only to a certain relative precision, known as the “machine epsilon” $\epsilon$. Here $\epsilon$ is the smallest positive real number such that, when calculated by the computer, $1 + \epsilon$ is found to be not equal to 1. (Beware that $\text{eps}$ in MATLAB uses a slightly different definition.) You will need to find out what $\epsilon$ is for the programming language and machine you are using: state its value in your write-up, and also show how you found it (or how you checked that a published value was correct).

From now on we shall assume that any real number stored or calculated in the computer is only accurate to within a relative error of $\epsilon$, so that when a number $x$ is input, the stored value actually represents the interval $(x_1, x_2)$, where $x_1 = x - \epsilon \times x$ and $x_2 = x + \epsilon \times x$; the true value of $x$ lies in that range. When we use an inbuilt operation (such as addition, division, etc.) or an intrinsic function ($\exp(x)$, $\sqrt{x}$, etc.) this inaccuracy may be compounded with a further relative error; for example, if in a program $y$ is set equal to $\exp(x)$, we can assume that the true value of $e^x$ lies in the range $(y_1 - \epsilon \times y_1, y_2 + \epsilon \times y_2)$ where $y_1 = \exp(x_1)$ and $y_2 = \exp(x_2)$.

(This works because $\exp$ is an increasing function.)

**Question 7** Write a program to develop the continued fraction expansion of a real number $x$, giving the partial quotients and convergents, as far as is justified. Your program should stop as soon as it is not possible to be sure of the next partial quotient.

Use your program to tabulate the partial quotients (and in a few cases the convergents) for the real numbers $\pi$, $2\pi$, $\pi^2$, $e - 1$, $3e^2$, $\tan(1/2)$, $\sqrt{5}\tan(1/\sqrt{5})$, $(9 + 2\sqrt{39})/15$ and $\sqrt{d}$ for $d = 1, \ldots, 20$ and $d = 180, \ldots, 200$. You will need to keep track of accumulated errors, as suggested above, remembering to give details in your report.

Comment on any structure you see in these continued fractions expansions. In some cases you may be able to guess how the expansions continue. Comment on finding a good rational approximation to $\pi$, $2\pi$ and $\pi^2$.

**References**
