University of Cambridge
Faculty of Mathematics

Mathematical Tripos
Part III Guide to Courses 2014-2015

The Faulkes Institute of Geometry, completed in January 2002

Mathematical Tripos
Part III Lecture Courses in 2014-2015

Department of Pure Mathematics & Mathematical Statistics

Department of Applied Mathematics & Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*, and are indicated as such after the title. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2014-15. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures and associated course materials as offered in this academic year define the syllabus. Each course lecturer has discretion to vary the material covered.
- Some courses have no writeup available at this time, in which case you will see "No description available" in place of a description. Course descriptions will be added to the online version of the guide to courses as soon as they are provided by the lecturer. Until then, the descriptions for the previous year (available at http://www.maths.cam.ac.uk/postgrad/mathiii/) may be helpful in giving a rough idea of course content, but beware of the comments in the preceding item on what defines the syllabus.

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Algebra

Commutative Algebra (M24)

C.J.B. Brookes

The aim of the course is to give an introduction to the theory of commutative Noetherian rings and modules, a theory that is an essential ingredient in algebraic geometry, algebraic number theory and representation theory.

Topics I hope to fit in will be the theory of ideals for Noetherian and Artinian rings; localisations and completions; integral closure, valuation rings and Dedekind rings; dimension theory; differential operators

Pre-requisites

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields will be helpful but not necessary.

Literature

1. M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra, Addison-Wesley, 1969.
2. N. Bourbaki, Commutative algebra, Elements of Mathematics, Springer, 1989 .
3. I. Kaplanksy, Commutative rings, University of Chicago Press, 1974.
4. H. Matsumura, Commutative ring theory, Cambridge Studies 8, Cambridge University Press, 1989.
5. M.Reid, Undergraduate Commutative Algebra, LMS student texts 29, Cambridge University Press, 1995.
6. R.Y. Sharp, Steps in commutative algebra, LMS Student Texts 19, Cambridge University Press, 1990.

   The basic text is Atiyah and Macdonald but it doesn't go into much detail and many results are left to the exercises. Sharp fills in some of the detail but neither book goes far enough. Both Kaplanksy and Matsumura cover the additional material though Matsumura is a bit tough as an introduction. Reid's book is a companion to one on algebraic geometry and that influences his choice of topics and examples. Bourbaki is encyclopaedic.

Additional support

Four examples sheets will be provided, with supporting examples classes.

Lie algebras and their representations (M24)

David Stewart

Lie algebras were introduced by Sophus Lie as a way to study what we now call Lie Groups. The latter can be thought of as smooth groups. Then Lie algebras arise by looking at infinitesimal transformations, specifically, of the tangent space at the identity. We'll go through these concepts in some detail, but actually the definition of a Lie algebra (which will be given in approximately three lines) is simply a vector space with a certain anticommutative multiplication which satisfies some version of associativity. So for the most part, all the geometry of the Lie group can be exorcised and we can get down to the algebraic arguments which will give us a complete picture of the finite-dimensional complex representations of finite-dimensional semisimple Lie algebras. But we'll do more than that, giving a classification of the complex simple Lie algebras by root data, covering all the structure theory necessary to get us there.

Lie theory comes in many flavours and is important in finite group theory (with 26 exceptions all nonabelian finite simple groups come from Lie theoretic objects), number theory (notably the Langlands programme), physics (e.g. quantum), differential equations, integrable systems ... Underpinning all Lie theoretical objects are root systems. In some way this course can be seen as an introduction to those most fundamental of mathematical objects, as motivated by Lie algebras.

Desirable Previous Knowledge

You need to be happy with the notion of a vector space but that's more-or-less it. I'm planning to illustrate many of the theorems by showing how they go wrong over fields of positive characteristic, so a basic familiarity with the existence of such fields would be good. Having taken some course on representation theory in the past would be a plus, only so that terms like 'completely reducible' are familiar.

Reading to complement course material

1. Representation theory, Fulton and Harris. Springer. This is a beautiful book written in a fun, chatty style with plenty of examples, motivation, and pictures. It tells a good story. It is the main source of the lecture notes and would be a great complement to the course. It also has stuff on representations of the symmetric groups. If you are thinking of staying on in algebra, it would be a great purchase.
2. Introduction to Lie algebras and representation theory. Humphreys. Springer. A good book, taking a more algebraic approach.
3. Introduction to Lie algebras. Erdmann and Wilson. Springer. Again more algebraic while a little more accessible than the Humphreys.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Representation Theory (L24)

Stuart Martin

The course deals with the representation theory of algebras and the related area of "quiver" representations. It is based on work begun in the 1960s by Gabriel, Auslander and several Soviet mathematicians, including Nazarova and Roiter. Quivers are very simple mathematical objects: finite directed graphs. A representation of a quiver assigns a vector space to each vertex, and a linear map to each arrow. Quiver representations were originally introduced to treat problems of linear algebra, for example, the classification of tuples of subspaces of a prescribed vector space. But it soon turned out that quivers and their representations play an important role in representation theory of fnite-dimensional algebras; they also occur in less expected domains of mathematics including Kac-Moody Lie algebras, quantum groups, Coxeter groups, and geometric invariant theory. This course presents some fundamental results on representations of quivers, from an algebraic and a geometric point of view. Our main goal is to give an account of a theorem of Gabriel characterizing quivers of finite orbit type, that is, having only finitely many isomorphism classes of representations in any prescribed dimension: such quivers are exactly the disjoint unions of Dynkin diagrams of types A_n, D_n, E_6, E_7, E_8, equipped with arbitrary orientations. Moreover, the isomorphism classes of indecomposable representations correspond bijectively to the positive roots of the associated root system. This beautiful result has many applications to problems of linear algebra.

Gabriel's theorem holds over an arbitrary field; in the course, we will only consider algebraically closed fields, in order to keep the prerequisites at a minimum. **Allergy warning**: the course contains some geometry!

- Quivers, representations, path algebras; examples
- Module and cohomology theory
- Review/introduction to algebraic groups;
- The representation variety
- Introduction to Euclidean and Dynkin diagrams
- Representation type, Gabriel's Theorem.
- Representations of finitely-generated algebras
- Research topic (if time permits): maybe Donkin's work from [3] below.

Two sheets of examples will be provided backed up by two or three classes.

Desirable Previous Knowledge

Prerequisites are fairly modest: basic notions about rings and modules; a little homological algebra (up to and including Ext^1 and the long exact sequence); some algebraic geometry (Zariski topology on affine space, dimension, morphisms, Zariski tangent spaces, differentials, varieties, affine schemes from [4], [5]); basic category theory.

Introductory Reading

1. M. Auslander, I. Reiten and S. O. Smalo, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Math. 36 (CUP, 1995).
2. D. J. Benson, Representations and cohomology I: Basic representation theory of finite groups and associative algebras, Cambridge Studies in Advanced Math. 30, (CUP, 1991).
3. S. Donkin, Polynomial invariants of representations of quivers, Comment. Math. Helv. 69 (1994), 137–141
4. D. Eisenbud, Commutative algebra with a view towards algebraic geometry, Graduate Texts in Math. 150, Springer-Verlag, New York (1995).
5. D. Eisenbud, J. Harris, The geometry of schemes, Graduate Texts in Math. 197, Springer-Verlag, New York (2000).
6. H. Kraft and Ch. Riedtmann, Geometry of representations of quivers, In *Representations of algebras* (P.J. Webb (ed)), LMS Lectures Note Series 116, CUP (1986).

Reading to complement course material

1. H. Derksen and J. Weyman, Quiver representations, Notices Amer. Math. Soc. 52 (2005), 200–206.
2. P. Etingof *et al*, Introduction to representation theory (chapter 6), Student Math. Library vol. 89 (AMS 2011).
3. M. Geck, An introduction to algebraic geometry and algebraic groups, Oxford Graduate Texts in Mathematics (OUP, 2003).
4. C. Ringel, Four papers on problems in linear algebra, in: Representation theory, 141–156, London Math. Soc. Lecture Note Ser. 69, (CUP, 1982).
5. T. A. Springer, Linear Algebraic Groups, Second edition, Progress in Math. 9, (Birkhauser, Basel, 1998).

Homological and Homotopical Algebra (L16)

Julian Holstein

This course is an introduction to homological and homotopical algebra, which are ubiquitous in modern algebra, algebraic topology, geometry and number theory. The aim is to learn to use these methods in whichever field you are working in.

We start with a look at chain complexes and homology, the derived category and derived functors. Early examples are Ext, Tor and group cohomology. Next we learn one of the most important computational tools, spectral sequences, and work through some applications.

The last part of the course makes the switch to homotopical algebra: We define model categories and consider examples from algebra and topology as well as their interplay.

Time permitting there are many other topics this course could visit along the way, like simpilical methods or Hochschild homology, but they will only be decided upon closer to the time.

Pre-requisites

This course assumes some familiarity with rings and modules and the language of categories and functors.

A course on algebraic topology is very useful, as such a course provides exposure to basic homological algebra and algebraic topology is one of the most fruitful fields of application. Similar comments apply to commutative algebra.

There will be examples from a range of subjects, but as long as students are willing to take some results on trust, no expertise is required.

Literature

1. W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories.* In Handbook of Algebraic Topology, 1995. Available at hopf.math.purdue.edu/Dwyer-Spalinski/theories.pdf

2. S. I. Gelfand and Yu. I. Manin, *Methods of Homological Algebra.* Springer, 2003.

3. C. Weibel, *An introduction to homological algebra.* Cambridge University Press, 1995.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Infinite-Dimensional Lie Algebras (L16)

Non-Examinable (Graduate Level)

Alexandre Bouayad

The aim of this course is to give an introduction to Kac-Moody algebras and their representations, to present some applications of the theory, and to have a look at the quantum counterpart.

Kac-Moody algebras are infinite-dimensional analogs of semisimple Lie algebras, and we will first study the structure of these algebras in general. The first significant examples will be the affine Lie algebras. These algebras have another realisation; namely they are central extensions of loop algebras, and therefore have important applications in theoretical physics. We will precise the relations with the Virasoro and the Heisenberg algebras, and we will study the category of finite-dimensional representations of an affine Lie algebra.

By construction, an affine algebra has a central element, which then acts as a scalar (called level) on every simple representation. We will look at a certain category formed by representations with fixed level, describe the fusion product within this category, and present applications to the Knizhnik-Zamolodchikov equations.

Quantum groups (or to be more precise, quantized enveloping algebras in this course) are quantum analogs of semisimple Lie algebras, and more generally of Kac-Moody algebras. If time permits, we will define these objects, study their representation theory (with an emphasis again on the affine case, which should lead us to the so-called q-characters), and discusss the relations between representations of classical and quantum affine algebras.

Pre-requisites

Essential: Lie algebras and their representations

Useful: Representation theory

Literature

1. V. Chari and A. Pressley, A guide to quantum groups, Cambridge University Press, Cambridge, 1995.
2. P. Etingof, I. Frenkel and A. Kirillov, Lectures on representation theory and Knizhnik-Zamolodchikov equations, Mathematical Surveys and Monographs, 58, American Mathematical Society, Providence, RI, 1998.
3. E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, Second edition, Mathematical Surveys and Monographs, 88, American Mathematical Society, Providence, RI, 2004.
4. E. Frenkel, Langlands correspondence for loop groups, Cambridge Studies in Advanced Mathematics, 103, Cambridge University Press, Cambridge, 2007.
5. J. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics, 9, Springer-Verlag, New York-Berlin, 1978.
6. J. C. Jantzen, Lectures on Quantum Groups, Graduate Studies in Mathematics, 6, American Mathematical Society, Providence, RI, 1996.
7. V. Kac, Infinite-dimensional Lie algebras, Cambridge University Press, Cambridge, 1990.
8. J-P. Serre, Lie algebras and Lie groups, 1964 lectures at Harvard University, Lecture Notes in Mathematics, 1500, 2006.

Analysis

Analysis of Partial Differential Equations (M24)

Clément Mouhot

The purpose of this course is to introduce some techniques and methodologies in the mathematical treatment of Partial Differential Equations (PDE). The theory of PDE is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th century. It is at a crossroad with physics and many areas of pure and applied mathematics.

The course begins with an introduction to four prototype linear equations: Laplace's equation, the heat equation, the wave equation and Schrödinger's equation. Emphasis will be given to the modern functional analytic techniques relying on the notion of Cauchy problem and estimates rather than explicit solutions, although the interaction with classical methods (e.g. the fundamental solution, Fourier representations) will be discussed. The following basic unifying concepts will be studied: well-posedness, energy estimates, elliptic regularity, characteristics, propagation of singularities, maximum principle. The course will end with a discussion of some of the open problems in PDE.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background; in particular a familiarity with measure and integration theory is useful. The course will be mostly self-contained and can be used as a first introductory course in PDE for students wishing to continue with some specialised PDE Part III courses in lent and easter terms (elliptic PDE, kinetic PDE, PDE and image processing. . .). In particular having attended the "Partial differential equations" course in Part II is useful but is *not* a pre-requisite.

Literature

Some lecture notes are available online at

`http://cmouhot.wordpress.com/teachings/`.

The following textbooks are excellent references:

1. Evans, L. C., *Partial Differential Equations*, Springer, 2010.
2. John, F., *Partial Differential Equations*, Springer, 1991.

The following review gives an overview of the field of PDE: Klainerman, S., *Partial Differential Equations*, Princeton Companion to Mathematics (editor T. Gowers), Princeton University Press, 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Functional Analysis (M24)

András Zsák

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. We begin with a review of some of the material of the Part II Linear Analysis course which will be taken for granted (see pre-requisites below). We then cover the following topics:

Hahn–Banach Theorem on the extension of linear functionals. Locally convex spaces.

Riesz Representation Theorem.

Weak and weak-* topologies. Theorems of Mazur, Goldstein, Banach–Alaoglu.

Hahn–Banach Theorem on separation of convex sets. Extreme points and the Krein–Milman theorem.

Banach algebras, spectral theory. Commutative Banach algebras and the Gelfand representation theorem. Holomorphic functional calculus.

Hilbert space operators, C^*-algebras. The Gelfand–Naimark theorem. Spectral theorem for commutative C^*-algebras. Spectral theorem and Borel functional calculus for normal operators.

Some additional topics time permitting.

Pre-requisites

Thorough grounding in basic topology and analysis.

Some knowledge of basic functional analysis. Specifically, the following results will be taken for granted (although, they will be recalled and, for some, proofs will be given): definition and examples of normed spaces and bounded linear operators; operator norm; equivalence of norms on finite-dimensional normed spaces; finite-dimensional subspace of normed space is closed; Baire Category Theorem, Open Mapping Lemma, Open Mapping Theorem, Closed Graph Theorem; Stone–Weierstrass Theorem; Urysohn's Lemma; Arzelà–Ascoli theorem. Hilbert spaces; orthogonal decompositions; orthonormal bases; Riesz Representation Theorem (the one identifying the dual of Hilbert space); adjoint operators.

In the section on the Riesz Representation Theorem and in Spectral Theory as well as in examples, some knowledge of measure theory will be very useful.

In Spectral Theory we will make use of basic complex analysis, for example, Cauchy's Integral Formula, Maximum Modulus Principle.

Literature

The first two books are excellent both for introductory reading and for the course. If you want to brush up on your measure theory, I really like the third book but, of course, there are plenty of others on the subject.

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales.* Oxford University Press, 2011. ISBN: 9780199206537, 9780199206544. Series title: Oxford graduate texts in mathematics 20. Available in Betty & Gordon Moore Library (QA322.2.A45 2011).

2. Bollobás, Béla *Linear analysis : an introductory course.* Cambridge University Press, 1990. ISBN: 0521383013, 0521387299. Available in Betty & Gordon Moore Library (QA320.B65 1990) and University Library South Front, Floor 4 (349:4.c.95.416).

3. Taylor, S. J. *Introduction to measure and integration.* Cambridge University Press 1973. ISBN: 0521098041. Available in Betty & Gordon Moore Library (QA312.T39 1973) and University Library South Wing, Floor 5 (202.c.97.337).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Analysis of Operators (M24)

Non-Examinable (Graduate Level)

Antony Wassermann

Starting from the spectral theorem for compact selfadjoint operators on Hilbert space, this course will study operators that occur in different parts of analysis, including partial differential equations, group representation theory, complex function theory and complex dynamical systems. This part of operator theory plays an important role in Teichmuller theory and conformal field theory. Topics will include:

- Sobolev spaces and elliptic regularity, Fredholm & Toeplitz operators, Conformal welding.
- Heisenberg group. Fourier transform. Stonevon Neumann theorem.
- SU(1, 1), SL(2,R) and SL(2,C). Oscillator representation and semigroup.
- Singular integral operators. Hilbert transform on T, R and C. Cauchy transform. Uniformisation of multiply connected domains.
- Analysis on regular and limit sets of Schottky groups. Poincare series. Szego projections. Transfer operators. Hausdorff measure and dimension. Dynamical zeta function.

Desirable Previous Knowledge

This course will assume knowledge of undergraduate courses in analysis, including complex analysis and functional analysis..A previous version of the course, covering all the above material except the last segment, can be found at

`http://www.dpmms.cam.ac.uk/~ajw`

Literature

1. G.B. Folland, Harmonic analysis in phase space.
2. E.M. Stein and R. Shakarchi, Real Analysis.
3. S.R. Bell, The Cauchy transform, potential theory, and conformal mapping.
4. L.V. Ahlfors, Lectures on quasiconformal mappings.
5. D. Mumford, C. Series and D. Wright, Indras Pearls: The Vision of Felix Klein.
6. K. Deimling, Nonlinear functional analysis.

Dispersive PDEs (L24)

Non-Examinable (Graduate Level)

Shiwu Yang

This course is aimed to provide a general frame work to study the local and global existence theory for several evolution equations, including Schodinger equations, wave equations and possibly Einstein equations in general relativity. We hope this course is self-contained. Hence we need to cover the necessary analysis tools in order obtain the local or global existence results. This may requires a general fixed point argument, distribution theory, Sobolev spaces and Sobolev embedding, Fourier transforms, interpolations, maximal functions, isoperimetric inequalities and so on. Then we will use these basic tools to study the classical local and global existence theory of nonlinear Schodinger and nonlinear wave equations. If time permitted, we will also discuss Einstein's equation in general relativity from the mathematical point of view.

Pre-requisites

This course assumes you know integration by parts.

Literature

1. T. Tao, *Nonlinear dispersive equations: Local and global analysis.* . Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006.

2. E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press, 1993.

3. C. Sogge, *Lectures on nonlinear wave equations.* International Press, Boston, 1995.

4. L. Evans, *Partial differential equations.* Graduate Studies in Mathematics, 19. American Mathematical Society,Providence, RI, 1998.

Semigroups of operators (L24)

Dr D.J.H. Garling

We begin with the basic theory of semigroups of operators, and then restrict attention to Feller semigroups, which are semigroups of positive linear operators on a space of continuous functions. All these semigroups are concerned with evolution in time, and so they occur naturally in mathematical physics and in the theory of stochastic processes. Examples include the heat semigroup, the Poisson semigroup, Brownian motion and other Markov semigroups, such as the Ornstein-Uhlenbeck semigroup. Important concepts here are notions of energy and entropy, wihich are used to study large deviations.

Desirable Previous Knowledge

Basic knowledge of analysis and general topology. Results from the the Part II Linear Analysis course will be used, as will results from the Part II Probability and Measure course, but detailed knowledge of their proofs will not be needed. Familiarity with Markov chains and Markov processes will be an advantage, and Dr Zsak's Michaelmas term course.

Introductory Reading

1. B. Bollobas. Linear Analysis. CUP 1990.

2. R.M.Dudley. Real analysis and probability (Chapters 1-5). CUP 2002.

3. J.R. Norris. Markov chains CUP 1997

Reading to complement course material

1. Jerome A. Goldstein. Semigroups of linear operators and applications. OUP 1985.

2. E Hille and R.S Phillips. Functional analysis and semigroups. AMS 1957.

3. E.B.Davies. One parameter semigroups. LMS 1980.

4. N. Berestycki and R.Nickl. Concentration of measure. On Dr Nickl's website. 2009.

5. M. Ledoux. The concentration of measure phenomenon. AMS 2001.

6. C.Ané et al. Sur les inégalités de Sobolev logarithmiques. SMF 2000.

Topics in Kinetic Theory (L24)

Amit Einav

Kinetic equation are a particular type of, usually non linear, Partial Differential Equations (PDEs) that arise in Statistical Physics. Their goal is to describe the time evolution of systems consisting of large amount of objects, such as Plasmas, Galaxies and Dilute Gases. This course is an introductory course to the modern analysis of kinetic equations, aiming to present some results on the fundamentally important Boltzmann equation from the subject of gas dynamics.
The course is suitable for both Pure Mathematics and Applied Mathematics students. I plan to cover the following topics:

1. Introduction:
   - Microscopic, Macroscopic and Mesoscopic Viewpoints and Kinetic Theory.
   - From ODEs to PDEs.
2. Derivation of Kinetic Equations:
   - Newtonian and Statistical Viewpoints.
   - The Characteristic Method.
   - The Many Particle Limit and Mean Field Models.
3. Linear Transport Equations:
   - Lagrangian and Eulerian Viewpoints.
   - Dispersion Estimations.
   - Averaging Lemma and Phase Mixing.
4. The Linear Boltzmann Equation:
   - A Probabilistic Interpretation.
   - The Cauchy Theory.
   - The Maximum Principle.
   - Relaxation to Equilibrium.

Pre-requisites

Knowledge of basic Measure Theory, Functional Analysis and simple methods in Ordinary Differential Equations (as in the 1A course 'Differential Equations') is required. Any advanced knowledge in the above topics, as well as knowledge in PDEs, Sobolev spaces and Fourier Analysis, can benefit the student, but is not mandatory. Students are welcome to discuss any pre-requisite requirements with the Lecturers prior to the beginning of the course.

Literature

The course is mainly self contained and requires no textbook, lecture notes will be provided. Also, there are numerous textbooks that will compliment the material of the course, or help bring the student up to pace with the pre requisites of it.

Additional support

Four examples sheets will be provided as well as four associated examples classes.

Elliptic Partial Differential Equations (L24)

Costante Bellettini and Brian Krummel

This course is intended as an introduction to the theory of elliptic partial differential equations. Elliptic equations play an important role in geometric analysis and a strong background in linear elliptic equations provides a foundation for understanding other topics including minimal submanifolds, harmonic maps, and general relativity. We will discuss both classical and weak solutions to elliptic equations, considering when solutions to the Dirichlet problem exist and are unique and considering the regularity of solutions. This involves establishing maximum principles, Schauder estimates, and other estimates on solutions. As time permits, we will discuss other topics including the De Giorgi-Nash theory, which can be used to prove the Harnack inequality and establish Hölder continuity for weak solutions, and quasilinear elliptic equations.

Pre-requisites

Lebesgue integration, Lebesgue spaces, Sobolev spaces, and basic functional analysis.

Literature

1. David Gilbarg and Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order. Springer-Verlag (1983).
2. Lawrence Evans, Partial Differential Equations. AMS (1998)
3. Qing Han and Fanghua Lin, Elliptic partial differential equations. Courant Lecture Notes, Vol. 1 (2011).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Introduction to nonlinear wave equations (L24)

Jonathan Luk

We will discuss the local and global theories for quasilinear wave equations and their applications to physical theories including fluid mechanics and general relativity. The following topics will be covered:

1. Quantitative behaviour of solutions to the linear wave equation in Minkowski spacetime
2. Energy methods and the local theory for quasilinear wave equations
3. Application in general relativity: local well-posedness of the Einstein equations
4. Examples of subcritical nonlinear wave equations
5. The null condition and the small-data global theory for quasilinear wave equations
6. Application in fluid mechanics: formation of shocks in spherical symmetry
7. Application in general relativity: stability of the Minkowski spacetime

Pre-requisites

Some exposure to partial differential equations, Fourier analysis and differential geometry will be useful but we will develop most of the necessary tools within course.

Literature

We will not follow any specific texts but students may find the literature listed below useful:

1. H. Ringström, *Non-linear wave equations*, available at

   `http://www.math.kth.se/~hansr/nlw.pdf`

2. C. Sogge, *Lectures in nonlinear wave equations* 2nd edition. International Press, 2011.
3. S. Klainerman, *Lecture notes in analysis*, available at

   `https://web.math.princeton.edu/~seri/homepage/courses/Analysis2011.pdf`

Part of the course will also be drawn from the following (recent) research papers:

1. H. Lindblad and I. Rodnianski, *Global stability of Minkowski spacetime in harmonic gauge*, Annals of Math (2010), also available at

   `http://arxiv.org/pdf/math/0411109v2.pdf`

2. G. Holzegel, S. Klainerman, J. Speck and W. Wong, *Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations: An Overview*, preprint, available at

   `http://arxiv.org/pdf/1407.6276v1.pdf`

Additional support

Examples sheets will be provided and example classes will be given.

Mean Curvature Flow and related topics (L16)

Non-Examinable (Graduate Level)

Neshan Wickramasekera

The course will provide an introduction to the theory of Mean Curvature Flow and related quasilinear PDE theory. The focus will be on the relevant regularity and singularity theories.

Pre-requisites

Knowledge of linear second order elliptic and parabolic PDE theory as well as some familiarity with the geometry of submanifolds of the Euclidean space will be very useful.

Literature

1. Klaus Ecker, *Regularity Theory for Mean Curvature Flow.* Birkhäuser, 2004.
2. Tom Ilmanen, *Singularities of mean curvature flow of surfaces.* http://www.math.ethz.ch/˜ ilmanen/papers/sing.ps, 1995.
3. Kota Kasai & Yoshihiro Tonegawa, *A general regularity theory for weak mean curvature flow.* Calc. Var. (2014), 50: 1-68.

Function Spaces (E20)

Non-Examinable (Graduate Level)

Sophia Demoulini

The course will provide an introduction to various useful classes of function spaces and provide examples to illustrate their use in the calculus of variations. The topics covered will be taken from the following, depending upon audience background and preference, and the time available.

Function Spaces

Review of pre-requisites: Measure Theory and Lebesgue spaces, Riesz representations on spaces of continuous functions, Egorov, Lusin BV spaces in 1 dimension.

Hardy-Littlewood Principle Calderon-Zygmund decomposition Weak and Strong (p,q) operators Covering Theorems (Vitali, Besicovich) Absolute Continuity Differentiation Hahn and Lebesgue Decomposition

Introduction to Fractional Integral Operators and the Hardy-Littlewood-Sobolev theorem. Sobolev Spaces and, possibly, spaces of BV in n-dimensions, to lead also to some theory in minimisation in BV spaces.

Pre-requisites

Basic analysis, including measure theory, and functional analysis.

Literature

1. Measure Theory and Fine Properties of Functions , Evans L.C. and Gariepy, R.F., CRC press (1991).
2. Direct Methods in the Calculus of Variations, Dacorogna B., Springer (2007).

Combinatorics

Combinatorics (M16)

Andrew Thomason

Combinatorics is used here in its traditional specific sense of combinatorial set theory, involving the study of families $\mathcal{A}$ of subsets of some finite set. What can be said about $\mathcal{A}$ if its members satisfy some special property, often specified in terms of size, intersection or containment? It is the nature of the subject that many natural and innocuous-sounding questions are difficult to address, and it is not possible to get to grips with them in a course of this kind. Nevertheless, the basic approaches that we introduce are already enough to make suprising progress.

At least the following topics should be treated. *Antichains.* The inequalities of Sperner and LYM. *Shadows.* How many subsets must be contained in members of $\mathcal{A}$? The Kruskal-Katona theorem shows the way. *Intersecting systems.* The Erdős-Ko-Rado theorem is the first piece of information about $\mathcal{A}$ if every pair of members intersect. *Exact intersections.* This refers to properties of the kind $|A \cap B| \equiv t \pmod{m}$. Theorems of Fisher and Frankl-Wilson have striking applications.

Pre-requisites

Some basic familiarity with discrete mathematics is assumed — for example, the theorems of Hall and of Ramsey.

Literature

1. Bollobás, B. Combinatorics, CUP (1986).

Additional support

It is expected that a couple of example sheets will be provided, and associated examples classes will be offered, together with a revision class in the Easter Term.

Techniques in Combinatorics (M16)

W. T. Gowers

Many of the best results in combinatorics are important not so much for the results themselves – though these can be very appealing – as for the techniques that have been introduced to prove them, which can often be used, or modified, to solve many other problems. The focus of this course will be on arguments with this generalizable quality. For each technique, I shall present a problem that would seem very hard to somebody who does not know the technique, but that becomes straightforward once one does know it. Thus, the aim of the course will be to turn people who attend it into more powerful mathematicians.

With this aim in mind, I shall try to present as many techniques as possible without going into any one of them too deeply, since usually it is sufficient to see one relatively simple example of a technique in order to recognise its potential applications to other problems. The following is a first approximation to what I shall cover.

1. Use of Markov's inequality and averaging arguments.
2. Use of the Cauchy-Schwarz inequality and the second-moment method.
3. Discrete Fourier analysis and applications.

4. Quasirandom sets and quasirandom graphs.
5. Szemerédi's regularity lemma and applications.
6. Dependent random selection.
7. Plünnecke's theorem.
8. Ruzsa's embedding lemma and related results.
9. Bogolyubov's method.
10. Use of Dirichlet's "box" principle.
11. Covering lemmas.
12. The Croot-Sisask lemma and applications.
13. The polynomial method and applications.
14. Use of the finite-dimensional Hahn-Banach theorem.

Although the emphasis will be on techniques, that does not mean that the course will not contain some memorable results. Among the results that I hope to use as illustrations are the Szemerédi-Trotter theorem, Roth's theorem for arithmetic progressions of length 3, the triangle removal lemma, the Balog-Szemerédi theorem, Freiman's theorem, lower bounds on the lengths of arithmetic progressions in sumsets, and the solution to the Kakeya problem for finite fields. Also, some results will be presented through examples sheet questions with suitable hints.

Pre-requisites

There are very few prerequisites. I will assume that you have seen the Cauchy-Schwarz inequality and know what a graph is. However, more important than any prior knowledge will be an enthusiasm for solving problems with elementary statements using not quite elementary methods.

Additional support

There will be two examples sheets with an examples class for each one. There may be a one-hour revision class in the Easter Term.

Probablistic Combinatorics (L16)

Béla Bollobás

Combinatorics is a young branch of mathematics driven by problems: the aim is to solve attractive and basic problems by any method available – in fact, the more sophisticated the method the better. Over the years, several methods have been introduced, although there is a long way to go before the combinatorialists will have a toolbox comparable to those in the well-established branches of mathematics.

The aim of the course is to present some of the methods that have been created so far, together with a number of their applications. The foremost of these methods is the *probabilistic method* introduced and popularized by Paul Erdős over fifty years ago: much of the course will be concerned with recent results concerning this method. In addition to this, the course will touch on algebraic, topological and Fourier analytical methods as well.

Pre-requisites

The course will be accessible to people knowing a smattering of probability theory and combinatorics, not higher than the Part II level. The lectures will not follow a book, but printed notes will be distributed about some of the material to be presented.

Literature

1. N. Alon, Combinatorial Nullstellensatz, in *Recent Trends in Combinatorics* (Mátraháza, 1995), *Combin. Probab. Comput.* **8** (1999), 7–29.

2. N. Alon and J.H. Spencer, *The Probabilistic Method*, Third edition, with an appendix on the life and work of Paul Erdős, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2008. xviii+352 pp.

3. P. Balister and B. Bollobás, Projections, entropy and sumsets, *Combinatorica* **32** (2012), 125–141.

4. I. Bárány, Geometric and combinatorial applications of Borsuk's theorem, in *New Trends in Discrete and Computational Geometry*, Algorithms Combin. **10**, Springer, Berlin, 1993, 235–249,

5. A. Björner, Topological methods, in *Handbook of Combinatorics* **1**, Elsevier, Amsterdam, 1995, pp. 1819–1872.

6. B. Bollobás, *Random Graphs*, Second edition, Cambridge Studies in Advanced Mathematics, **73**, Cambridge University Press, Cambridge, 2001, xviii+498 pp.

7. B. Bollobás and A. Thomason, Projections of bodies and hereditary properties of hypergraphs, *Bull. London Math. Soc.* **27** (1995), 417–424.

8. J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial, The influence of variables in product spaces, *Israel J. Math.* **77** (1992), 55–64.

9. P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in *Infinite and Finite Sets* (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, pp. 609–627.

10. L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *Journal of Combinatorial Theory*, Series A, **25** (1978), 319–324.

11. M. Talagrand, Concentration of measure and isoperimetric inequalities in product spaces, *Inst. Hautes Études Sci. Publ. Math.* No. 81 (1995), 73–205.

12. M. Talagrand, Transportation cost for Gaussian and other product measures, *Geom. Funct. Anal.* **6** (1996), 587–600.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in Ergodic Theory (L24)

Péter Varjú

Ergodic theory studies dynamical systems on which an invariant measure is given. There are many examples of such systems that originate from other branches of mathematics. This led to a fruitful interplay between ergodic theory and other fields. One of the earliest examples of this is Furstenberg's proof of Szemerédi's theorem.

Szemerédi's theorem asserts that any subset of the integers of positive density contains an arbitrarily long arithmetic progression. Furstenberg observed that this statement can be expressed in terms of recurrence properties of certain dynamical systems, and he gave an ergodic theoretic proof.

I will explain some basic elements of ergodic theory, such as recurrence, ergodic theorems, mixing and joinings. I will finish the course with the above mentioned proof of Furstenberg.

Desirable Previous Knowledge

Measure theory and basic functional analysis.

Reading to complement course material

Einsiedler, Ward, *Ergodic Theory with a view towards Number Theory*, Springer, 2011.

Topics in Random Graphs (L16)

Non-Examinable (Graduate Level)

Lutz Warnke

Having its origins in work of Erdős, the classical theory of random graphs was developed systematically by Erdős and Rényi in the 1960s. Since then the theory has advanced tremendously, using a blend of ideas and techniques from combinatorics, probability theory and statistical physics, say.

This course discusses some selected modern aspects of the theory and of the methods involved. Particular emphasis will be given to the *interpolation method* (a statistical physics method for showing that the mean of renormalized random variables converges to a deterministic limit), *Talagrand's inequality* (a concentration of measure methodology for showing that random variables are typically close to their mean), and the algorithmic *Lovász Local Lemma* (yielding a randomized algorithm which, intuitively speaking, can quickly find a needle in a haystack). If time permits, we might also briefly discuss the phenomenon of phase transitions.

Pre-requisites

We shall only assume some basic notions of probability and graph theory.

Literature

1. N. Alon and J. Spencer. *The Probabilistic Method.* John Wiley & Sons Inc. (2008).
2. M. Bayati, D. Gamarnik and David, P. Tetali. Combinatorial approach to the interpolation method and scaling limits in sparse random graphs. *Ann. Probab.* **41** (2013), 4080-4115.
3. B. Bollobás. *Random Graphs*, Cambridge University Press (2001).
4. S. Boucheron, G. Lugosi and P. Massart. *Concentration inequalities.* Oxford University Press (2013).
5. S. Janson, T. Łuczak and A. Ruciński. *Random Graphs.* Wiley-Interscience, New York (2000).
6. R. Moser and G. Tardos. A constructive proof of the general Lovsz local lemma. *J. ACM* **57** (2010), 15 pp.

Geometry and Topology

Algebraic Topology (M24)

Jacob Rasmussen

Algebraic topology assigns algebraic invariants (groups and homomorphisms) to topological spaces and continuous maps between them. The most important example of such an invariant is ordinary homology theory, which is part of the basic language of geometry today. This course will cover homology and cohomology, together with applications to the topology of manifolds and vector bundles. The emphasis will be on learning to compute and use these invariants in a variety of examples. A tentative syllabus is as follows:

- *Homology.* Singular homology and cohomology. Eilenberg-Steenrod axioms and cellular homology. Universal coefficient theorem. Künneth theorem and cup products.
- *Topology of Manifolds.* Handle decompositions and Morse theory. Poincaré duality. The Lefshetz fixed point theorem.
- *Vector Bundles.* Vector bundles and principal bundles. The Thom isomorphism and the Euler class. Long exact sequence on homotopy groups. Classifying spaces for bundles.

Pre-requisites

The only required background is basic point-set topology, but prior experience with the fundamental group would be helpful. The material in the Michaelmas term Differential Geometry course will be useful as well.

Literature

1. R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer (1982).
2. A. Hatcher, *Algebraic Topology*, CUP (2002).
3. J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press (1999).
4. J. W. Vick, *Homology Theory*, Springer (1994).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Algebraic Geometry (M24)

P.M.H. Wilson

This will be a basic course introducing the tools of modern algebraic geometry, and applying them to deduce (for instance) the Riemann–Roch theorem for smooth projective curves. The most relevant reference for the course is the book of Kempf.

Topics to be covered are sheaves, abstract varieties (over an algebraically closed field) and their properties, coherent sheaves, divisors, sheaf cohomology, differentials and the Riemann–Roch Theorem. I shall not introduce schemes, but the proofs I'll give will be in such a style that there are natural extensions to the case of schemes.

Pre-requisites

Basic theory on rings and modules will be assumed. Students will find it helpful to have looked beforehand at the book on Commutative Algebra by Atiyah and MacDonald, and/or the elementary text by Reid on Algebraic Geometry.

Literature

Introductory Reading

1. M. Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press (1988) (preliminary reading).
2. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison–Wesley (1969) (basic text also for the commutative algebra we'll need).

Reading to complement course material

1. G.R. Kempf, *Algebraic Varieties*, Cambridge University Press (1993) (main reference).
2. R. Hartshorne, *Algebraic Geometry*, Springer (1977) (more advanced text).
3. I. Shafarevich, *Basic Algebraic Geometry*, Springer (1974) (useful background reading).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Written answers to each examples sheet will be available a couple of days before the associated examples class.

Differential Geometry (M24)

A. Kovalev

This course is intended as an introduction to modern differential geometry. It can be taken with a view of further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are e.g. in Analysis or Mathematical Physics. Tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, matrix Lie groups. Tangent vectors, tangent and cotangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Exterior algebra of differential forms. Orientability of manifolds. Partition of unity and integration on manifolds, Stokes' Theorem. De Rham cohomology.
- *Vector Bundles.* Structure group, principal bundles. The example of Hopf bundle. Bundle morphisms. Three views on connections: vertical and horizontal subspaces, Christoffel symbols, covariant derivative. The curvature form and second Bianchi identity.
- *Riemannian Geometry.* Connections on manifolds, torsion. Riemannian metrics, Levi–Civita connection. Geodesics, exponential map, Gauss' Lemma. Decomposition of the curvature of a Riemannian manifold, Ricci and scalar curvature, low-dimensional examples. The Hodge star and Laplace–Beltrami operator. Statement of the Hodge decomposition theorem (with a sketch-proof, time permitting).

Pre-requisites

Essential pre-requisite is a working knowledge of linear algebra (including dual vector spaces/dual maps and bilinear forms) and of multivariate calculus (in particular, differentiation in several variables and the inverse function theorem). The course will not assume previous knowledge of manifolds.

Preliminary Reading

Students might like to read some of Chapter 1 in [3] or some of [4] in advance.

Literature

[1] R.W.R. Darling, *Differential forms and connections.* CUP, 1994.

[2] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry.* Springer-Verlag, 1990.

[3] V. Guillemin, A. Pollack, *Differential topology.* Prentice-Hall Inc., 1974.

[4] M. Spivak, *Calculus on manifolds.* W.A. Benjamin Inc., 1965.

[5] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, 1983.

Roughly, half of the course material is taken from [5]. The book [3] covers the required topology. On the other hand, [1], which has a chapter on vector bundles and on connections, assumes no knowledge of topology. Both [1] and [2] have a lot of worked examples. There are many other good differential geometry texts.

Additional support

The lectures will be supplemented by four example classes. Printed notes will be available from
`www.dpmms.cam.ac.uk/~agk22/teaching.html`

Complex Manifolds (L24)

David Witt Nyström

The goal is to help students learn the basic theory of complex manifolds. A preliminary outline of the course is as follows.

- Basic concepts of complex manifolds, holomorphic vector bundles, holomorphic tangent and cotangent bundles (for which corresponding concepts from the real smooth manifolds will be assumed). Canonical line bundles, normal bundle for a submanifold and the adjunction formula.
- Brief description of sheaf cohomology, with deduction of de Rham and Dolbeault cohomology for complex manifolds.
- Hermitian metrics, connections, curvature and Chern classes for complex vector bundles. Case of holomorphic vector bundles.
- Harmonic forms: the Hodge theorem and Serre duality (general results on elliptic operators will be assumed).
- Compact Kähler manifolds. Hodge and Lefschetz decompositions on cohomology, Kodaira-Nakano vanishing, Kodaira embedding theorem.

Pre-requisites

A knowledge of basic Differential Geometry from the Michaelmas Term course will be highly desirable.

Literature

1. J. P. Demailly *Complex analytic and differential geometry.* Available as pdf at

   `http://www-fourier.ujf-grenoble.fr/~demailly/documents.html`

2. P. Griffiths and J. Harris *Principles of Algebraic Geometry.* Wiley, 1978.
3. D. Huybrechts *Complex Geometry - an introduction.* Springer, 2004.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Homotopy Theory (L24)

Oscar Randal-Williams

This will be an advanced course in Algebraic Topology, introducing two of the most important computational tools in this subject: spectral sequences and cohomology operations. After developing the basic theory we will focus on calculations and applications of these tools.

Topics that might be covered are

1. the spectral sequence of a filtered space, the Serre and bar spectral sequences, example calculations,
2. cohomology of classifying spaces of classical groups, relation to characteristic classes,
3. Steenrod operations and the Steenrod algebra,
4. Eilenberg–MacLane spaces, Postnikov systems, and obstruction theory,
5. the method of killing homotopy groups, some homotopy groups of spheres.

Pre-requisites

Part III Algebraic Topology.

Literature

1. J. Strom *Modern Classical Homotopy Theory.* Graduate Studies in Mathematics, 127.
2. A. Hatcher *Algebraic Topology.* Cambridge University Press, 2002. Available at `http://www.math.cornell.edu/∼hatcher/AT/AT.pdf`
3. R. E. Mosher and M. C. Tangora *Cohomology operations and applications in homotopy theory.* Dover.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in Algebraic Geometry (L24)

Mark Gross

This course is intended to be a follow-up of the Part III Algebraic Geometry class offered during Michaelmas term. It will begin by covering some of the more advanced topics in Hartshorne's book on the subject; in particular, we will begin with an introduction to schemes. We will then explore the geometry of curves and surfaces in more depth. Topics will include the Deligne-Mumford moduli spaces of stable curves, the classification of surfaces, and the minimal model program for surfaces.

Pre-requisites

The minimal prerequisite is the Part III course in Algebraic Geometry.

Literature

1. R. Hartshorne, *Algebraic Geometry.* Springer-Verlag, 1977.

Geometric group theory (L24)

Henry Wilton

The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts. This course will provide an introduction to the basic ideas of the subject. We will focus on the contrast between the behaviour of *arbitary* finitely presented groups, which can be very badly behaved in the sense that they exhibit algorithmic undecidability, and *generic* finitely presented groups, which turn out to be well behaved and amenable to study. We will also emphasize the connections with differential geometry on the one hand and with algorithmic questions on the other.

Part 1. We will introduce the basic notions of geometric group theory: Cayley graphs, quasi-isometries, the Schwarz–Milnor Lemma, and the connection with algebraic topology via presentation complexes. We will discuss the word problem, which is quantified using the Dehn functions of a group.

Part 2. We will deepen the connection with geometry with a discussion of the Filling Theorem, which relates the Dehn function of the fundamental group of a Riemannian manifold M to the solutions of Plateau's Problem in M.

Part 3. We will discuss the algorithmic aspects of groups. We will exhibit groups with unsolvable word problem, and state Higman's embedding theorem, the Novikov–Boone theorem etc.

Part 4. This is the technical heart of the course. We will cover the basic theory of word-hyperbolic groups, including the Morse lemma, local characterization of quasigeodesics, linear isoperimetric inequality, finitely presentedness, quasiconvex subgroups etc.

Pre-requisites

Part IB Geometry and Part II Algebraic topology are required.

Literature

1. P. de la Harpe, *Topics in geometric group theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000. vi+310 pp.

2. J. Meier, *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups.* London Mathematical Society Student Texts, 73. Cambridge University Press, Cambridge, 2008. xii+231 pp.

3. M. R. Bridson and A. Haefliger *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999. xxii+643 pp.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Spectral Geometry (L24)

Non-Examinable (Graduate Level)

Dennis Barden

Although the roots of this subject go back further, at least to Gelfand's conjecture that the length spectrum of a Riemann surface determines it up to isometry, it was popularised by Kac's 1966 paper titled 'can one hear the shape of a drum' in which he asked whether the spectrum of the Laplacian on a compact domain with smooth boundary in the plane determines that domain up to isometry. The answer to that is that we still do not know. However failure to answer that question naturally spurred its generalisation to arbitrary Riemannian manifolds where much progress has been made in the intervening 50 years.

Unsurprisingly, to my naive mind, there the answer 'no' was provided by the construction of pairs, and even continuous families, of manifolds that are isospectral yet are not isometric. On the other hand these examples are so special, usually highly symmetric and, in the first examples, having a common Riemannian covering that it is still possible to conjecture that generically, in an appropriate sense, Riemannian manifolds are indeed determined by their Laplacian spectrum. Certainly their dimension and volume are so determined.

In this graduate version of the Part III course that I have given for a number of years I aim to skim more lightly over that material to allow time to discuss in more depth:

- the generalisations and extensions of Sunada's covering space method for producing counterexamples, particularly the work of Gordon and Schueth generalising it to torus bundles and the interpretation of it, by Bérand and Buser, as the 'transplantation' of eigen-functions and geodesics from one manifold to another.
- work in the direction of generic determination, particularly Colin de Verdière's on the generic equivalence of the length and Laplacian spectra and Wolpert's result that isospectral but non-isometric Riemann surfaces must lie in a proper analytic subvariety of the Teichmüller space.

As the course is not yet written I cannot guarantee the extent to which I may achieve those aims!

Pre-requisites

The subject is distinctly polymathic requiring aspects of geometry, analysis and topology, as well as algebra and other topics. However, rather than being a master of all these areas, it will suffice to be a jack-of-all-trades who is willing to accept the relevant definitions and results at face value, usually with appropriate references.

Literature

The best, relatively brief, introductory survey that I know is

- C.S.Gordon, Survey of Isospectral Manifolds, Handbook of Differential Geometry, Vol I, pp 747-778, Elsevier Science, 2000.

Supporting material

I shall make available LaTexed notes covering rather more than the previous Part III courses and will also attempt to expand them to cover this course.

Topics in algebra and geometry (L24)

Non-Examinable (Graduate Level)

Ian Grojnowski

I'll choose one or two topics that are currently of interest in algebraicgeometric approaches to representation theory and develop them, explaining the basic examples as well as the necessarymachinery surrounding them.

One possible topic is W-algebras and their generalisations.

We would begin with the Beilinson-Bernstein theorem, which describes representations of a Lie algebra $\mathfrak{g}$ in terms of D-modules on the flag variety. One of the first consequences of this theorem is the Kazhdan-Lusztig conjecture, which describes the characters of irreducible highest weight modules for $\mathfrak{g}$.

I'll then explain what a W-algebra is, and describe their representations. This is work of many people including Bezrukavnikov, Ostrik, Losev, Premet, Ginzburg, McGerty and Nevins.

Pre-requisites

No particular background will be assumed other than the standard language of algebraic geometry, and a first course on Lie algebras, though some homological algebra will also be useful.

Quantum Cohomology (L24)

Non-Examinable (Graduate Level)

Ivan Smith

This will be a graduate-level topics course. The (small) quantum cohomology ring $QH^*(X)$ of a symplectic manifold X is a deformation of the usual cohomology ring involving counts of rational curves in X (three-point Gromov-Witten invariants). This course will briefly discuss some background material on the analysis of holomorphic curve theory, explain the definition of quantum cohomology (in a simple case, say for monotone symplectic manifolds), and will then cover some of the following topics: explicit computations, applications to four-dimensional symplectic topology, the Seidel representation and Hamiltonian group actions, Frobenius manifolds, integrable systems. Towards the end of the course we may widen the focus to discuss one or more of symplectic cohomology, Lagrangian Floer cohomology or connections to mirror symmetry.

Pre-requisites

Familiarity with basic algebraic topology, differential geometry and symplectic topology at least. The level of sophistication of different sections of the course will probably vary quite a lot, as will the level of detail given and the amount of background presumed.

Literature

References to papers for specific topics we look at will be given in the course. For general background and the basic development of the theory:

1. D. McDuff and D. Salamon. *J-holomorphic curves and quantum cohomology.* AMS 1994.

2. D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology.* AMS 2004.

Topics on Complex Geometry (E16)

Non-Examinable (Graduate Level)

Julius Ross

This is a graduate course on complex geometry. The topic will be pluripotential theory leading up to the complex Homogeneous Monge Ampère equation. We will discuss the relevance of this to space of Kähler potentials, and discuss connection with certain free boundary problems.

Pre-requisites

The course will assume good knowledge of the theory of complex manifolds.

Literature

1. J-P Demailly *Complex analytic and differential geometry.* Available at

   `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`

2. S Donaldson *Holomorphic Discs and the Complex Mounge-Ampère equation* J. Symplectic Geom. Volume 1, Number 2 (2002), 171–196.

3. S Donaldson *Nahm's equations and the classification of monopoles* Comm. Math. Phys. Volume 96, Number 3 (1984), 387–407.

4. J Ross and D Witt Nyström *Harmonic Discs of Solutions to the Complex Homogeneous Monge-Ampère Equation* arXiv:1408.6663

Logic

Category Theory (M24)

Dr. R.B.B. Lucyshyn-Wright

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in approximately the first three-quarters of the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories.

Representable functors. The Yoneda lemma. Representations of functors.

Limits. Construction of limits from products and equalizers. Preservation and creation of limits.

Monomorphisms and epimorphisms. Strong, regular, and split mono- and epimorphisms.

Adjunctions. Equivalent definitions of the notion of adjunction. Uniqueness of adjoints. Reflections and coreflections. The Adjoint Functor Theorems.

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions.

The remainder of the course will be devoted to topics chosen by the lecturer, probably from among the following:

Filtered colimits. Finitary functors, finitely-presentable objects. Applications to universal algebra.

Abelian categories. Kernels and cokernels. Additive categories. Image factorisation in abelian categories. Exact sequences. Introduction to homological algebra.

Monoidal categories. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

Fibrations. Fibred and indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Pre-requisites

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. Mac Lane, S. *Categories for the Working Mathematician*, Springer 1971 (second edition 1998). A classic one-volume book on the subject, written by one of its founders.

2. Awodey, S. *Category Theory*, Oxford U.P. 2006. A more gently-paced treatment.

3. Borceux, F. *Handbook of Categorical Algebra*, Cambridge U.P. 1994. Three volumes which together provide a self-contained account of subjects seen as requisite knowledge for research in category theory and its applications. Volume 1 covers most but not all of the Part III course.
4. McLarty, C. *Elementary Categories, Elementary Toposes* (chapters 1–12 only), Oxford U.P. 1992. A gently-paced introduction to categorical ideas, demanding little mathematical background.

Additional support

Four examples sheets will be provided and four one-hour examples classes will be given. There will be a one-hour revision class in the Easter Term.

Model Theory (L16)

Benedikt Löwe

Model theory is the subfield of mathematical logic that studies the interplay between syntactic properties of theories and properties of the corresponding classes of structures. Typically, it deals with first-order logic, the system of logic that is usually taken as the framework in which we formalize all of our mathematical discourse. As is proved in an introductory mathematical logic course (such as the Part II course *Logic and Set Theory*), first-order logic satisfies the *Completeness Theorem* which has serious repercussions for the expressive power of first-order logic.

This course starts from this observation, revisiting the consequences of the Completeness Theorem for first-order logic and considering some alternatives (such as infinitary logics). After this, we shall study various central constructions for the study of model theory: game-theoretic representations of the expressive power of logics (Ehrenfeucht-Fraïssé theory), quantifier elimination, realizing and omitting types, and ultraproducts.

The course (in some cases in combination with *Topics in Set Theory*) provides the background for a number of essay topics on infinite games and ultrapower constructions in set theory.

Pre-requisites

This course builds on the Part II course *Logic and Set Theory.* The Part III course *Topics in Set Theory* (Michaelmas Term 2014) is not a pre-requisite, but will provide additional background understanding during parts of the course.

Literature

1. C. C. Chang and H. J. Keisler, *Model Theory.* Third edition. Studies in Logic and the Foundations of Mathematics, Vol. 73. North-Holland Publishing Co., Amsterdam, 1990.
2. H.-D. Ebbinghaus, J. Flum, and W. Thomas, *Mathematical logic.* Second edition. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1994.
3. W. Hodges, *A shorter model theory.* Cambridge University Press, Cambridge, 1997.
4. D. Marker, *Model theory. An introduction.* Graduate Texts in Mathematics, Vol. 217. Springer-Verlag, New York, 2002.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Topics in Set Theory (M24)

Oren Kolman

This course is a relatively self-contained introduction to independence results in modern set theory and their repercussions in contemporary mathematics. It focuses on the ideas and techniques in the proofs, using forcing, that the Continuum Hypothesis ($2^{\aleph_0} = \aleph_1$) and combinatorial assertions relating to infinite trees can be neither proved nor refuted from the principles of ordinary set theory. Applications in algebra, analysis and topology will be illustrated. We shall treat several of the following topics.

Infinitary combinatorics. Cofinality. Stationary sets. Fodor's lemma. Filters and ideals. Ulam's theorem. Cardinal exponentiation. Beth and Gimel functions. Generalized Continuum Hypothesis. Singular Cardinals Hypothesis. Partial orders and trees: Aronszajn, Suslin, and Kurepa. Prediction principles (diamonds, squares). Martin's Axiom. Hypotheses of Suslin and Kurepa; the tree property and weak compactness.

Axiomatics. The formal axiomatic system of ordinary set theory (ZFC). Models of set theory. Absoluteness. Simple independence results. Transfinite recursion. Ranks. Reflection principles. Constructibility.

Forcing. Generic extensions. The forcing theorems. Examples. Adding reals; collapsing cardinals. Introduction to iterated forcing. Internal forcing axioms. Proper forcing.

Large cardinals. Introduction to large cardinals. Ultrapowers. Scott's theorem. Embeddings of V.

Partition relations and possible cofinality theory. Partition relations. Model-theoretic methods. Ramsey's theorem; Erdős-Rado theorem. Kunen's theorem. Walks on ordinals. Todorcevic's theorem. Introduction to pcf theory.

Pre-requisites

The Part II course *Logic and Set Theory* or its equivalent is essential.

Literature

Basic material

1. † Drake, F. R., Singh, D., *Intermediate Set Theory*, John Wiley, Chichester, 1996.
2. Eklof, P. C., Mekler, A. H., *Almost Free Modules*, rev. ed., North-Holland, Amsterdam, 2002.
3. Halbeisen, L., *Combinatorial Set Theory With a Gentle Introduction to Forcing*, Springer, Berlin, 2012.
4. Kanamori, A., *The Higher Infinite*, 2^{nd} ed., Springer, Berlin, 2009.
5. † Kunen, K., *Set Theory*, reprint, Studies in Logic, 34, College Publications, London, 2011.
6. Schindler, R., *Set Theory - Exploring Truth and Independence*, Universitext, Springer, 2014.

Advanced topics

7. Burke, M. R., Magidor, M., Shelah's pcf theory and its applications, Ann. Pure Appl. Logic 50 (1990), 207–254.
8. Kanamori, A., Foreman, M., *Handbook of Set Theory*, Springer, Berlin, 2012.
9. † Shelah, S., *Proper and Improper Forcing*, 2^{nd} ed., Springer, Berlin, 1998. Chapters 1 and 2.
10. Shelah, S. *Cardinal Arithmetic*, Oxford University Press, New York, 1994.
11. Todorcevic, S., Combinatorial dichotomies in set theory, Bull. Symbolic Logic 17 (2011), 1–72.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. Individual consultations will be offered. There will be a two-hour revision class in the Easter Term.

Computability and Logic (M24)

Dr Thomas Forster

Recursive datatypes. Structural and wellfounded induction. Finite state machines: regular expressions, regular languages, the pumping lemma and Kleene's theorem. Primitive Recursive functions. The justification of inductive definition. Ackermann and n-recursive functions. General Recursive functions. Turing Machines. Lambda-representable functions, Curry-Howard. Semidecidable and decidable sets. Unsolvability of the Halting problem. Rice's theorem. Recursive inseparability and Tennenbaum's theorem. Automatic structures (automatic groups) and automatic theories. Recursive structures. Recursive saturation. Recursive ordinals and hierarchies of fast-growing functions. Scott's isomorphism theorem. Kolmogoroff Complexity (set of incompressible strings is immune). Axiomatisable and nonaxiomatisable theories: Craig's theorem, Trakhtenbrod's theorem. Incompleteness of arithmetic. Undecidability of Predicate calculus. Introductory Degree theory: Kleene-Post, Friedberg-Muchnik, Baker-Gill-Solovay.

Pre-requisite Mathematics

The course is designed to be the sequel to Part II Logic and Set Theory, so no specific competance in computability is assumed or required.

Literature

There are numerous textbooks with titles like this course, and I can't think of any that the prospective reader needs to be warned against. Two suitable CUP books easily available locally with your student discount are:

G. Boolos and R. Jeffrey "Computability and Logic" CUP paperback

N Cutland "Computability" CUP paperback

Earlier editions of Boolos-and-Jeffrey are to be preferred to the latest version prepared by Burgess. Mendelson's *Introduction to Mathematical Logic* is a good general background.

Number Theory

Local Fields (M16)

Tom Fisher

The theory of local fields was introduced by Hensel in the early 1900's as an alternative approach to algebraic number theory. The basic idea is to consider the completions of a number field K at all absolute values, not just the ones arising from the embeddings of K into the reals or complexes. One can then borrow techniques from analysis to study K and its finite extensions in a way that focuses on their behaviour at just one prime. For instance the analogue of the Newton-Raphson method for root finding goes by the name of Hensel's lemma.

Nowadays, local fields have established themselves as a natural tool in many areas of number theory and also in subjects like representation theory, algebraic topology and arithmetic geometry. More immediately, the course will serve as preparation for the "Algebraic Number Theory" course.

The course will begin by introducing the field of p-adic numbers $\mathbb{Q}_p$ (where p is a prime). This is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic absolute value defined for non-zero $x \in \mathbb{Q}$ by $|x|_p = 1/p^n$ where $x = p^n a/b$ with p not dividing a or b. Topics to be covered will then include: absolute values on fields, valuations, complete fields and their extensions, inverse limits, Hensel's lemma and ramification theory. If time permits, then possible further topics include: local class field theory (statements only), the Hilbert norm residue symbol, and the Hasse-Minkowski theorem.

Pre-requisites

Basic algebra up to and including Galois theory is essential. Some prior exposure to algebraic number fields is recommended.

Literature

1. J.W.S. Cassels, *Local fields*, CUP, 1986.
2. G.J. Janusz, *Algebraic number fields*, AMS, 1996.
3. N. Koblitz, *p-adic numbers, p-adic analysis and zeta-functions*, Springer, 1977.
4. J. Neukirch, *Algebraic number theory*, Springer, 1999.
5. J.-P. Serre, *A course in arithmetic*, Springer, 1973.
6. J.-P. Serre, *Local fields*, Springer, 1979.

Additional support

There will be three example sheets and three associated examples classes.

Topics in Number Theory (M24)

Non-Examinable (Graduate Level)

Wansu Kim

F-crystals with G-structure and Affine Deligne-Lusztig "varieties"

To any projective complex manifold, one can associate (integral) Hodge structures via classical Hodge theory. The virtue is that Hodge structures are completely linear algebraic in nature. Analogously, to any proper smooth variety over a perfect field of characteristic p, one can associate certain semi-linear algebraic objects (so-called F-crystals) by the theory of crystalline cohomology.

Affine Deligne-Lusztig "varieties" are conjectural schemes (in characteristic p) which classify F-crystals with some extra group-theoretic properties. (The inverted commas are because the representability in scheme is not known in the full generality, although it is known in many cases.) They naturally arise when one attempts to describe, in purely group-theoretic terms, all abelian varieties with extra structure up to isomorphism (or more generally, mod p points of Shimura varieties). Although the motivation is from arithmetic geometry, affine Deligne-Lusztig varieties can be defined and studied purely group-theoretically.

In this course, we aim to give an overview of the theory of F-(iso)crystals with G-structure and affine Deligne-Lusztig varieties under the unramified assumption. We will review necessary backgrounds in group theory along the way. If time permits, we will discuss some applications to mod p geometry of moduli of abelian varieties with extra structure. The contents of the course will be mostly group-theoretic.

Pre-requisites

This course assumes Part III level of algebraic number theory and algebraic geometry. Some basics on reductive groups will be reviewed during the course when needed.

Literature

I do not mean to cover all the materials in the references below.

1. M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Princeton University Press, 1996.
2. M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astérisque No. 298 (2005), 271–318
3. R. Kottwitz, *Isocristals with additional structure*, Compositio Math. **52** (1985), no. 2, 201–220.
4. R. Kottwitz, *Isocristals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339.
5. M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181.
6. R. Kottwitz and M. Rapoport, *On the existence of F-crystals*, Comment. Math. Helv. **78** (2003), no. 1, 153–184.
7. R. Kottwitz, *On the Hodge-Newton decomposition for split groups*, Int. Math. Res. Not. (2003), no. 26, 1433–1447.
8. X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, preprint.
   Available at

   `http://arxiv.org/abs/1407.8519`

Elementary Methods in Analytic Number Theory (L24)

Adam Harper

The classical approach of analytic number theory is to apply Cauchy's Residue Theorem to generating functions like the Riemann zeta function. Traditionally, methods avoiding this use of complex analysis were called "elementary". It was once thought that certain results were too deep to be proved in an elementary way, but this belief collapsed when Erdős and Selberg found an elementary proof of the Prime Number Theorem. Nowadays the strongest results are obtained by combining complex analysis with "elementary" methods of real analysis and combinatorics.

The course will cover some of the following topics, depending on time and audience preferences.

1. *Selberg's sieve.* The idea of sieving out primes. Problems with an obvious attempt at sieving. Selberg's sieve framework, and the optimal choice of the weights. Application to counting primes in short intervals and arithmetic progressions. Comparison with zeta function methods.

2. *Large sieve.* The large sieve as an approximate Bessel formula for exponential sums. The arithmetic formulation, and applications. Vaughan's identity for sums over primes. The multiplicative large sieve, and application to the average distribution of primes in progressions.

3. *Other sieve-flavoured arguments.* Selberg's fundamental lemma, and an elementary proof of the Prime Number Theorem. Application of Selberg-type weights to showing bounded gaps between primes.

4. *Linnik's dispersion method.* The difficulty of Goldbach-type additive problems with only two variables. Linnik's dispersion/variance framework. Some details of Linnik's treatment for sums of primes and binary quadratic forms.

Pre-requisites

This course has no formal pre-requisites. It would be helpful to have attended a first course on number theory, containing things like a statement of Bertrand's Postulate or the Prime Number Theorem, but this won't be assumed. The course will have a flavour of estimating complicated objects and handling error terms, which might be familiar from previous courses in analysis or probability.

Literature

1. H. Davenport, *Multiplicative Number Theory.* Third edition. Springer GTM vol. 74, 2000.

2. J. Friedlander and H. Iwaniec, *Opera de Cribro.* AMS Colloquium Publications vol. 57, 2010.

Friedlander and Iwaniec's book covers the Selberg and large sieves, and a huge amount of other material. Davenport's book has an excellent treatment of the large sieve and the application to primes in progressions on average (Bombieri–Vinogradov theorem), and is also a good introduction to the complex analysis parts of analytic number theory. The only book I know of that covers the dispersion method is Linnik's original monograph, *The Dispersion Method in Binary Additive Problems*, which is nice but is certainly not a textbook. There are many other analytic number theory books that cover at least some of the material, for example Montgomery and Vaughan, *Multiplicative Number Theory I. Classical Theory*, discusses the Selberg sieve and an elementary proof of the Prime Number Theorem.

Additional support

I plan to write three examples sheets and run three associated examples classes. There will also be a revision class in the Easter Term.

Algebraic Number Theory (L24)

Jack Thorne

Algebraic number theory lies at the foundation of much current research in number theory, from Fermat's last theorem to the proof of the Sato–Tate conjecture, and is a beautiful subject in its own right. This will be a second course in algebraic number theory, with an emphasis on the adele-theoretic (rather than the classical ideal-theoretic) point of view. We will assume familiarity with the basic notions of number fields and local fields, which will be reviewed briefly at the beginning of the course.

Topics likely to be covered include:

Decomposition of primes in extensions of number fields. Relation between local and global fields.

Adeles, ideles, and applications to class groups and groups of units.

Artin and abelian L–functions, and the analytic class number formula.

Chebotarev density theorem.

Pre-requisites

First courses in number fields and local fields (or equivalent reading) are recommended. A good understanding of Galois theory and basic algebra will be essential.

Literature

1. S. Lang, *Algebraic number theory.* Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994.
2. J. W. S. Cassels and A. Fröhlich, Eds., *Algebraic number theory.* 2nd edition. London Mathematical Society, 2010.
3. A. Weil, *Basic number theory.* Classics in Mathematics. Springer-Verlag, Berlin, 1995.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Probability

Advanced Probability (M24)

Alan Sola

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

Review of measure and integration: sigma-algebras, measures, and filtrations; integrals and expectation; convergence theorems; product measures, independence, and Fubini's theorem.
Conditional expectation: Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.
Martingales: Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.
Stochastic processes in continuous time: Kolmogorov's criterion, regularization of paths; martingales in continuous time.
Weak convergence: Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.
Sums of independent random variables: Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.
Brownian motion: Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.
Poisson random measures: Definitions, compound Poisson processes; Infinite divisibility, the Lévy-Khinchin formula, Lévy-Itô decomposition.

Prerequisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Literature

- D. Applebaum, Lévy processes (2nd ed.), Cambridge University Press 2009.
- R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
- O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.
- D. Williams, Probability with martingales, CUP 1991.

Additional support

Four example sheets will be provided along with supervisions. There will be at least one one-hour revision class in Easter term.

Stochastic Calculus and Applications (L24)

M. Tehranchi

This course is an introduction to the theory of continuous-time stochastic processes, with an emphasis on the central role played by Brownian motion. It complements the material in Advanced Probability, Advanced Financial Models, and Schramm–Loewner Evolutions.

- *Review of Brownian motion.* Isonormal process. Wiener's existence theorem. Sample path properties.
- *Continuous stochastic calculus.* Martingales, local martingales and semi-martingales. Quadratic variation and co-variation. Itô's isometry and definition of stochastic integral. Kunita–Watanabe's theorem. Itô's formula.
- *Applications to Brownian motion.* Lévy's characterization of Brownian motion. Dubins–Schwartz theorem. Girsanov's theorem. Transience and recurrence. Martingale representation theorems.
- *Stochastic differential equations.* Strong and weak solutions. Notions of existence and uniqueness. Yamada–Watanabe theorem. Strong Markov property. Kolmogorov, Fokker–Planck and Feynmann–Kac partial differential equations. The one-dimensional case. Stochastic partial differential equations.

Pre-requisites

Knowledge of measure theoretic probability at the level of Part III Advanced Probability will be assumed, especially familiarity with discrete-time martingales and basic properties of Brownian motion.

Literature

1. I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer. 1998
2. D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer. 2001
3. L.C. Rogers and D. Williams.*Diffusions, Markov Processes and Martingales. Vol.1 and 2.* Cambridge University Press. 2002

Additional support

Four sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Percolation and Related Topics (L16)

Geoffrey Grimmett and Demeter Kiss

The percolation process is the simplest probabilistic model for a random medium in finite-dimensional space. It has a central role in the general theory of disordered systems arising in the mathematical sciences, and it has strong connections with statistical mechanics. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to the theory of phase transitions in physics.

The basic theory of percolation will be described in this course, with some emphasis on areas for future development. The fundamental techniques, including correlation and/or concentration inequalities and their ramifications,will be covered. The related topics may include self-avoiding walks, and further models from interacting particle systems, and (if time permits) certain physical models for the ferromagnet such as the Ising and Potts models.

Pre-requisite Mathematics

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

The following texts will cover the majority of the course, and are available online.

1. Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2010; see http://www.statslab.cam.ac.uk/∼grg/books/pgs.html
2. Grimmett, G. R., *Three theorems in discrete random geometry*, Probability Surveys 8, (2011) 403–411; see http://arxiv.org/abs/1110.2395

Additional support

It is expected that there will be two example sheets and two classes.

Rough Paths and Regularity Structures (E12)

Non-Examinable (Graduate Level)

Peter Friz

Rough path analysis has provided new insights in the analysis of stochastic differential equations and stochastic partial differential equations. When applied to stochastic systems, rough path analysis provides a means to construct a pathwise solution theory which, in many respects, behaves much like the theory of deterministic differential equations and provides a clean break between analytical and probabilistic arguments. It provides a toolbox allowing to recover many classical results without using specific probabilistic properties such as predictability or the martingale property. The study of stochastic PDEs has recently led to a significant extension, Hairer's theory of regularity structures, and the second half of this course is devoted to a gentle introduction.

Pre-requisite Mathematics

Upper undergraduate analysis, interest (and a little maturity) in stochastic analysis.

Literature

1. Terry J. Lyons, Michael Caruana, and Thierry Levy, Differential equations driven by rough paths, Lecture Notes in Mathematics, vol. 1908, Springer, Berlin, 2007
2. Peter K. Friz, Nicolas Victoir, Multidimensional stochastic processes as rough paths, Cambridge Studies in Advanced Mathematics, vol. 120, Cambridge University Press, Cambridge, 2010
3. Peter K. Friz and Martin Hairer, A course on rough paths: With an introduction to regularity structures, Springer Universitext, 2014.
4. Martin Hairer, A theory of regularity structures, Inventiones mathematicae (2014), 1–236

Statistics

The courses in statistics form a coherent Masters-level course in statistics, covering theoretical statistics, applied statistics and applications. You may take all of them, or a subset of them. Core courses are Modern Statistical Methods and Applied Statistics in the Michaelmas Term.

All statistics courses for examination in Part III assume that you have taken an introductory course in statistics and one in probability, with syllabuses that cover the topics in the Cambridge undergraduate courses Probability in the first year and Statistics in the second year. It is helpful if you have taken more advanced courses, although not essential. For Applied Statistics and other applications courses, it is helpful, but not essential, if you have already had experience of using a software package, such as R or Matlab, to analyse data. The statistics courses assume some mathematical maturity in terms of knowledge of basic linear algebra and analysis. However, they are designed to be taken without a background in measure theory, although some knowledge of measure theory is helpful for Topics in Statistical Theory.

The desirable previous knowledge for tackling the statistics courses in Part III is covered by the following Cambridge undergraduate courses. The syllabuses are available online at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year		*Courses*
First	*Essential*	Probability
Second	*Essential*	Statistics
	Helpful for some courses	Markov Chains
Third	*Helpful*	Principles of Statistics
	Helpful for applied statistics courses	Statistical Modelling
	For additional background	Probability and Measure

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation. If you have more time, then it would be helpful to review other courses as indicated.

Modern Statistical Methods (M16)

Rajen Shah

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data where we have huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to study such datasets. We aim to cover a selection of the following topics:

- Penalised regression, including Ridge regression, the Lasso and variants;
- Machine learning methods including Boosting, Support vector machines, and the kernel trick;
- Multiple testing, including the False Discovery Rate and the Benjamini–Hochberg procedure;
- Graphical modelling and aspects of causal inference.

Pre-requisites

Basic knowledge of statistics, probability and linear algebra.

Literature

1. T. Hastie, R. Tibshirani and J. Friedman *The Elements of Statistical Learning.* 2nd edition. Springer, 2001.
2. P. Bühlmann, S. van de Geer, *Statistics for High-Dimensional Data.* Springer, 2011.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Applied Statistics (M+L24)

Susan Pitts, Davide Pigoli and Brian Tom

This course is split over two terms, with 16 hours (8 lectures and 8 practical classes) in the Michaelmas Term and 8 hours (4 lectures and 4 practical classes) in the Lent Term. It is a *practical* course aiming to develop skills in analysis and interpretation of data. In the Michaelmas Term, core topics in applied statistics are studied, while in the Lent Term more specialised topics are covered.

The statistical methods below will be put into practice using `R`.

Michaelmas Term

Introduction to `R`. Use of LaTeXfor report writing. Exploratory data analysis, graphical summaries.

Linear regression and its assumptions. Relevant diagnostics: residuals, Q-Q plots, leverages, Cook's distances and related methods. Hypothesis tests for linear models, ANOVA, F-tests. Interpretation of interactions.

The essentials of generalised linear modelling. Discrete data analysis: binomial and Poisson regression. Diagnostics, goodness-of-fit and model selection.

Lent Term

Some special topics. Previous examples include generalised additive models, and longitudinal data analysis.

Pre-requisites

It is assumed that you will have done an introductory statistics course, including: elementary probability theory; maximum likelihood; hypothesis tests (t-tests, χ^2-tests, F-tests); confidence intervals.

Literature

1. Dobson, A.J. and Barnett A. (2008) *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC.
2. Agresti, A. (1990) *Categorical Data Analysis.* Wiley.
3. McCullagh, P. and Nelder, J.A. (1989) *Generalized Linear Models.* Chapman & Hall.
4. Venables, W.N. and Ripley, B.D. (2002) *Modern Applied Statistics with S.* Springer-Verlag. 4th edition.
5. Pawitan, Y. (2001) *In All Likelihood: Statistical Inference Using Likelihood.* Oxford Science Publications.

Practical classes and additional support

This course includes practical classes in both the Michaelmas and Lent Terms, where statistical methods are introduced in a practical context and where students carry out analysis of datasets under the guidance of the lecturer. In practical classes, the students have the opportunity to discuss statistical questions with the lecturer.

For the core part of the course in the Michaelmas Term two examples sheets will be provided, and there will be two associated examples classes. Because emphasis in this course is placed on the importance of the clear presentation of statistical analyses, students will also have the opportunity to hand in written reports on two analyses, and these will be marked and feedback given to students. There will be a revision class in the Easter Term.

Actuarial Statistics (M16)

S.M. Pitts

This course provides an introduction to various topics in non-life insurance mathematics. These topics feature mainly in the Institute and Faculty of Actuaries examination CT6.

Topics covered in lectures include

1. Loss distributions
2. Reinsurance
3. Aggregate claims
4. Ruin theory
5. Credibility theory
6. No claims discount systems

Pre-requisite Mathematics

This course assumes

an introductory probability course (including moment generating functions, probability generating functions, conditional expectations and variances)

a statistics course (including maximum likelihood estimation, Bayesian methods)

that you know what a Poisson process is

that you have met discrete time finite statespace Markov chains

Literature

1. S. Asmussen and H. Albrecher *Ruin Probabilities.* 2nd edition. World Scientific, 2010.
2. C.D. Daykin, T. Pentikäinen and E. Pesonen, *Practical Risk Theory for Actuaries and Insurers.* Chapman and Hall, 1993.
3. D.M. Dickson, *Insurance Risk and Ruin.* CUP, 2005.
4. J. Grandell, *Aspects of Risk Theory.* Springer, 1991.
5. R.J. Gray and S.M. Pitts, *Risk Modelling in General Insurance: From Principles to Practice.* CUP, 2012.
6. T. Rolski, H. Schmidli, V. Schmidt and J. Teugels, *Stochastic Processes for Insurance and Finance.* Wiley, 1999.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Biostatistics (M10+L14)

This course consists of two components, Statistics in Medical Practice (10 lectures) and Analysis of Survival Data (14 lectures). Together these make up one 3 unit (24 lecture) course. You must take the two components together for the examination.

Statistics in Medical Practice (M10)

R. Turner, C. Jackson, J. Wason, J. Bowden, D. de Angelis, A. Presanis, P. Birrell, S. Seaman

Each lecture will be a self-contained study of a topic in biostatistics, which may include clinical trials, meta-analysis, missing data, multi-state models and infectious disease modelling. The relationship between the medical issue and the appropriate statistical theory will be illustrated.

Pre-requisites

Undergraduate-level statistical theory, including estimation, hypothesis testing and interpretation of findings.

Literature

There are no course books, but relevant medical papers will be made available before some lectures for prior reading. It would be very useful to have some familiarity with media coverage of medical stories involving statistical issues, e.g. from Behind the Headlines on the NHS Choices website: `http://www.nhs.uk/News/Pages/NewsIndex.aspx`. A few books to complement the course material are listed below.

1. Armitage P, Berry G, Matthews JNS. *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001
2. Borenstein M, Hedges L, Higgins JPT, Rothstein HR. *Introduction to Meta-Analysis.* Wiley, 2009
3. Jennison C, Turnbull B. *Group Sequential Methods with Applications to Clinical Trials.* Chapman and Hall, 2000

Additional support

Example sheets and solutions will be provided for each lecture topic.

Analysis of Survival Data (L14)

F. P. Treasure

Fundamentals of Survival Analysis:

Characteristics of survival data; censoring. Definition and properties of the survival function, hazard and integrated hazard. Examples.

Review of inference using likelihood. Estimation of survival function and hazard both parametrically and non-parametrically.

Explanatory variables: accelerated life and proportional hazards models. Special case of two groups. Model checking using residuals.

Current Topics in Survival Analysis:

In recent years there have been lectures on: frailty, cure, relative survival, empirical likelihood, counting processes and multiple events.

Pre-requisites

This course assumes assumes that you have attended an undergraduate course in statistics and that you are familiar with hypothesis testing, point and interval estimation, and likelihood methods. Attendance at the Michaelmas term course 'Applied Statistics' would be very helpful, not least for the introduction to the R language.

Literature

1. P. Armitage, J. N. S. Matthews and G. Berry *Statistical Methods in Medical Research* (4th ed.), Oxford: Blackwell (2001) [Chapter on Survival Analysis for preliminary reading].
2. D. R. Cox and D. Oakes *Analysis of Survival Data* London: Chapman and Hall (1984).
3. M. K. B. Parmar and D. Machin *Survival Analysis: A Practical Approach* Chichester: John Wiley (1995)
4. Therneau T.M. and Grambsch P.M. *Modelling Survival Data: Extending the Cox Model* New York: Springer (2000)

Additional support

There will be a two hour revision class based on student-selected examination questions in the Easter Term.

Topics in Statistical Theory (L16)

Richard Nickl

We shall rigorously cover a few selected topics in theoretical statistics, focusing on high- and infinite-dimensional (=nonparametric) statistical models. Topics will include:

→ Basic ideas of *nonparametric statistics*: density estimation, nonparametric regression, Gaussian white noise models

→ *Minimax theory in 'complex' statistical models*: basic information theory, lower bounds via Kullback-Leiber divergences, rates of convergence

→ *Wavelets and adaptive inference*: thresholding estimators, testing of multiple hypotheses

→ $p > n$*-problems and compressed sensing*: including a concentration of measure proof of the *restricted isometry property* for Gaussian sensing matrices.

Pre-requisites

Having seen a preliminary course in mathematical statistics (such as Part II Principles of Statistics) is certainly helpful but – at least for the mathematically gifted – not necessary. Elements of linear & functional analysis as well as probability & measure theory will feature in the proofs – background in these areas can be acquired as we go along (or beforehand). The Michaelmas Part III course on *Modern Statistical Methods* is also recommended.

Literature

Although we shall not explicitly follow it, the monograph

A. B. Tsybakov, *Introduction to nonparametric estimation*, Springer, 2009.

contains many of the relevant materials. There are also some lecture notes on *Statistical Theory* on the lecturers website that will be partially (but not entirely) relevant.

Additional support

Two example sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Time Series and Monte Carlo Inference (L24)

This course consists of two components, Time Series, and Monte Carlo Inference, each consisting of 12 lectures. Together these make up one 3 unit (24 lecture) course. You must take the two components together for the examination.

Time Series (L12)

Yining Chen

A time series is a sequence of observations collected through time. Time series data are regularly collected in many fields, including economics, finance and environmental sciences. This course will focus on the fundamental concepts required for the description, modelling and forecasting of time series data. An introduction to the theoretical foundation of time series models will also be provided. Topics to be covered include: descriptive methods, linear and non-linear time series models, tools for model identification and estimation, and basic spectral analysis.

Pre-requisites

Basic knowledge of statistics and probability.

Literature

1. P. J. Brockwell and R. A. Davis, *Introduction to Time Series and Forecasting*, 2nd edition. Springer Texts in Statistics, 2002. (Good introduction, especially for those completely new to time series.)

2. P. J. Brockwell and R. A. Davis, *Time Series: Theory and Methods*, 2nd edition. Springer Series in Statistics, 2009.

3. C. Francq and J-M. Zakoian, *GARCH Models: Structure, Statistical Inference and Financial Applications.* Wiley, 2010.

Additional support

Two examples sheets will be provided and two associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Monte Carlo Inference (L12)

Shahin Tavakoli

Monte Carlo methods are concerned with the use of stochastic simulation techniques for statistical inference. These have had an enormous impact on statistical practice, especially Bayesian computation, over the last 20 years, due to the advent of modern computing architectures and programming languages. This course covers the theory underlying some of these methods and illustrates how they can be implemented and applied in practice. The following topics will be covered: Techniques of random variable generation, Monte Carlo integration, Importance Sampling, Markov chain Monte Carlo (MCMC) methods for Bayesian inference, Gibbs sampling, Metropolis-Hastings algorithm, reversible jump MCMC.

Pre-requisites

You should have attended introductory Probability and Statistics courses. A basic knowledge of Markov chains would be helpful. Prior familiarity with a statistical programming package such as R or MATLAB would also be useful.

Literature

1. P. J. E. Gentle, *Random Number Generation and Monte Carlo Methods*, (Second Edition). Springer (2003).
2. B. D. Ripley, *Stochastic Simulation.* Wiley (1987).
3. W.D. Gamerman and H. F. Lopes, *Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference*, (Second Edition). Chapman and Hall (2006).
4. C.P. Robert and G. Casella, *Monte Carlo Statistical Methods*, (Second Edition). Springer (2004).
5. C. P. Robert and G. Casella, *Introducing Monte Carlo Methods with R*, Springer (2010).

Additional support

Two examples sheets will be provided and two associated examples classes will be given. There will be a revision class in the Easter Term.

Random Matrix Theory and High-dimensional Statistical Inference (M8)

Non-Examinable (Graduate Level)

Danning Li

Main part of this course is to give a survey of random matrix theory. We will explore different methods used for random matrix models. We will also discuss the application of the random matrix theory to the high-dimensional statistical inference.
The course will follow the agenda below:

(1) Introduction to random matrices:
  a) Popular models in random matrices.
  b) Eigenvalues and the entries of random matrices.

(2) Density functions of eigenvalues of some random matrices.

(3) Limit theorems for eigenvalues of popular random matrices:
  a) Semi-circular law.
  b) Machenko–Pastur law.
  c) CLT for Linear spectral statistics.
  d) Limits of the extreme eigenvalues and Tracy–Widom law.

(4) Applications to covariance structure testing:
  a) Likelihood ratio test for large n and large p.
  b) The limiting distribution of the off diagonal entries of sample correlation matrices.

Pre-requisites

Probability measure theory.

Literature

1. Z. Bai and J. Silverstein *Spectral analysis of large dimensional random matrices.* 2nd edition. Springer, 2009.
2. G. Anderson, A. Guionnet and Zeitouni, *An introduction to random matrices.* No. 118. Cambridge University Press, 2010.

Statistics for Stochastic Processes (L8)

Non-Examinable (Graduate Level)

Jakob Söhl

This course gives an introduction to inference for stochastic processes. Stochastic processes are widely used for modelling in many fields and are especially popular in finance. Different classes of processes are introduced and studied such as Lévy processes and Itô semimartingales. Observations schemes include high-frequency observations, low-frequency observations as well as observations given by option prices when the stochastic process models an asset price. Estimation procedures for the volatility, the drift and the jump measure are discussed in terms of rates, optimality and central limit theorems.

Pre-requisites

This course assumes knowledge of probability theory as covered by the lecture Advanced Probability in Cambridge, for example of the topic of martingales, the construction of Brownian motion and its properties. Knowledge of stochastic calculus will not be assumed. A familiarity with statistical concepts can be useful but is not a necessary prerequisite for the course.

Literature

1. Y. Aït-Sahalia and J. Jacod, *High Frequency Financial Econometrics.* Princeton University Press, 2014.

More literature will be provided along the course.

Operational Research and Mathematical Finance

Mathematics of Operational Research (M24)

Felix Fischer

The course covers a selection of mathematical tools and models for operational research:

- Lagrangian sufficiency theorem. Lagrange duality. Supporting hyperplane theorem. Sufficient conditions for convexity of the optimal value function. Fundamentals of linear programming. Linear program duality. Shadow prices. Complementary slackness. [2]
- Simplex algorithm. Two-phase method. Dual simplex algorithm. Gomory's cutting plane method. [3]
- Complexity of algorithms. NP-completeness. Exponential complexity of the simplex algorithm. Polynomial time algorithms for linear programming. [2]
- Network simplex algorithm. Transportation and assignment problems, Ford-Fulkerson algorithm, max-flow/min-cut theorem. Shortest paths, Bellman-Ford algorithm, Dijkstra's algorithm. Minimum spanning trees, Prim's algorithm. MAX CUT, semidefinite programming, interior point methods. [5]
- Branch and bound. Dakin's method. Exact, approximate, and heuristic methods for the travelling salesman problem. [3]
- Cooperative and non-cooperative games. Two-player zero-sum games. Existence and computation of Nash equilibria, Lemke-Howson algorithm. Bargaining. Coalitional games, core, nucleolus, Shapley value. Mechanism design, Arrow's theorem, Gibbard-Satterthwaite theorem, VCG mechanisms. Auctions, revenue equivalence, optimal auctions. [9]

Pre-requisites

The course is accessible to a candidate with mathematical maturity who has no previous experience of operational research; however it is expected that most candidates will already have had exposure to some of the topics listed above.

Literature

1. M.S. Bazaraa, J.J. Jarvis and H.D. Sherali: Linear Programming and Network Flows, Wiley (1988).
2. D. Bertsimas, J.N. Tsitsiklis. Introduction to Linear Optimization. Athena Scientific (1997).
3. N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani. Algorithmic Game Theory. Cambridge University Press (2007).
4. M. Osborne, A. Rubinstein: A Course in Game Theory. MIT Press (1994).

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Stochastic Networks (M24)

Frank Kelly

This course uses stochastic models to shed light on important issues in the design and control of communication networks. Randomness arises in communication systems at many levels: for example, the initiation and termination times of calls in a telephone network, or the statistical structure of the arrival streams of packets at routers in the Internet. How can routing, flow control and connection acceptance algorithms be designed to work well in uncertain and random environments? And can we design these algorithms using simple local rules so that they produce coherent and purposeful behaviour at the macroscopic level?

The first two parts of the course will describe a variety of classical models that can be used to help understand the performance of large-scale communication networks. Queueing and loss networks will be studied, as well as random access schemes. Parallels will be drawn with models from physics, and with models of traffic in road networks.

The third part of the course will more recently developed models of packet traffic and of congestion control algorithms in the Internet. This is an area of some practical importance, with network operators, hardware and software vendors, and regulators actively seeking ways of delivering new services reliably and effectively. The complex interplay between end-systems and the network has attracted the attention of economists as well as mathematicians and engineers.

Desirable previous knowledge

Mathematics that will be assumed to be known before the start of the course: Part IB Optimization and Markov Chains. Familiarity with Part II Applied Probability would be useful, but is not assumed.

Introductory reading

A feeling for some of the ideas of the course can be taken from

The mathematics of traffic in networks. In *Princeton Companion to Mathematics* (Edited by Timothy Gowers; June Barrow-Green and Imre Leader, associate editors) Princeton University Press, 2008. 862-870.

Literature

Reference 3 is the course text.

1. B. Hajek *Communication Network Analysis.*
2. F.P. Kelly *Reversibility and Stochastic Networks.* Cambridge University Press, 2011.
3. F. Kelly and E. Yudovina *Stochastic Networks.* Cambridge University Press, 2014.
4. R. Srikant and L. Ying *Communication Networks: An Optimization, Control and Stochastic Networks Perspective.* Cambridge University Press, 2013.

Additional support

Examples sheets will be provided and associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Advanced Financial Models (M24)

M. Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability and Stochastic Calculus & Applications.

- *Discrete time models.* Filtrations and martingales. Arbitrage, state price densities and equivalent martingale measures. Attainable claims and market completeness. European and American claims. Optimal stopping.
- *Brownian motion and stochastic calculus.* Brief survey of stochastic integration. Girsanov's theorem. Itô's formula. Martingale representation theorem.
- *Continuous time models.* Admissible strategies. Black–Scholes model. The implied volatility surface. Pricing and hedging via partial differential equations. Dupire's formula. Stochastic volatility models.
- *Interest rate models.* Short rates, forward rates and bond prices. Markovian short rate models. The Heath–Jarrow–Morton drift condition.

Pre-requisites

A knowledge of probability theory at the level of Part II Probability & Measure will be assumed. Familiarity with Part II Stochastic Financial Models is helpful.

Literature

1. M. Baxter & A. Rennie. *Financial calculus: an introduction to derivative pricing.* Cambridge University Press. 1996
2. M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling.* Springer. 2006
3. D. Kennedy.*Stochasic Financial models.* Chapman & Hall. 2010
4. D. Lamberton & B. Lapeyre.*Introduction to stochastic calculus applied to finance.* Chapman & Hall. 1996
5. S. Shreve. *Stochastic Calculus for Finance: Vol. 1 and 2.* Springer-Finance. 2005

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Optimal Investment (L16)

L. C. G. Rogers

The course will study a wide range of optimal investment/consumption problems that arise in theory and practice, and will discuss the usefulness of the conclusions. Most examples studied will be in a continuous-time setting. The following provisional list of topics indicates some of the intended content; not all the topics on this list will necessarily be covered, and topics may be covered that are not on this list.

- Self-financing portfolios and the wealth equation;
- the Merton problem and its solution in the CRRA case, using the Hamilton-Jacobi-Bellman approach;
- the Merton problem, general case, by martingale representation;
- the Merton problem, general case, using state-price density approach;

- (Davis-Varaiya) martingale principle of optimal control;
- dual methodology and the Pontryagin principle;
- equilibrium pricing;
- the equity premium puzzle;
- utility-indifference pricing;
- Lagrangian martingale characterisation of optimal solutions;
- imperfections: transaction costs, portfolio constraints, parameter uncertainty, infrequent rebalancing;
- varied objectives: ratcheting of consumption, habit formation, recursive utility;
- backward SDEs and optimal control;
- How good are any of these rules in practice?

Pre-requisites

A firm grasp of martingale theory, and a working knowledge of (Brownian) stochastic calculus will be required in the course.

Literature

1. I. Karatzas & S. E. Shreve: *Methods of Mathematical Finance*, Springer, 1998.

2. L. C. G. Rogers: *Optimal Investment*, Springer, 2013

Additional support

Two examples sheets will be provided and two associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Contest Theory (L16)

Dr. M. Vojnović

This course uses the game theory and statistics to provide theoretical underpinnings for the design of contests that arise in various real-life situations. A particular focus is devoted to the elements of contests that arise in the context of the design of the Internet e-commerce and online services. Broadly speaking, a contest is a system in which agents invest efforts in order to win one or more prizes. The goal of a contest owner is to maximize a given objective by using a suitable prize allocation mechanism. For example, the objective may be the total contribution or the best individual contribution solicited from the contestants, and the prize allocation mechanism may be allocating the prize to a contestant who exhibited the best performance or allocating in proportion to a measure of exhibited performance quality. The course focuses on the analysis of strategic equilibrium including the level of total contribution, maximum individual contribution, and social efficiency (price of anarchy). The theory of contests has been developed over years in the context of the economic theory, including public choice and political economy, the operations research, and more recently, computer science. The competition-based incentives have been used over centuries to solicit innovations and they constitute a significant part of the design of modern online services, e.g. the use of crowdsourcing platforms and referral incentives.

The course will cover some of the following topics:

1. Standard all-pay contest: complete information game, non-existence of pure-strategy Nash equilibrium, existence and full characterization of mixed-strategy Nash equilibria, exclusion principle, caps; incomplete information game, existence and uniqueness of a symmetric Bayes-Nash equilibrium, revenue equivalence.

2. Rank order allocation of prizes: complete information game and multiple prizes, multiple identical prizes; incomplete information game and conditions for optimality of rewarding a single prize, sensitivity on the shape of production cost functions, status prizes.

3. Smooth allocation of prizes: contest success functions, axiomatic and uncertainty justification of particular families of contest success functions, general logit contest success functions, existence and uniqueness of pure-strategy Nash equilibrium under general logit contest success functions, existence and characterization of pure-strategy Nash equilibrium under ratio-form, proportional, weighted proportional, and weighted valuation prize allocations, optimal contest success functions, difference-form contest success functions.
4. Simultaneous contests: complete information game of simultaneous standard all-pay contests, the Colonel Blotto game, incomplete information game of simultaneous standard all-pay contests, existence, uniqueness, full characterization of equilibrium, simultaneous contests with proportional prize allocation.
5. Sequential contests: equilibrium properties of contests with sequential allocation of prizes, comparison with single grand contest with simultaneous moves, multiple-round contests, sequential allocation of a prize with a termination rule.
6. Public goods: tragedy of commons, complete information game of utility sharing, egalitarian sharing, proportional sharing, smoothness framework for establishing price of anarchy bounds, the effect of production cost functions.
7. Tournaments: seeding of tournaments, standard seeding procedure, random permutation seeding, randomized cohort seeding, dynamic seeding procedures, desirable properties of delayed confrontation, monotonicity and envy freeness, strategic theory of tournaments, optimal prize split across stages.
8. Rating systems: probabilistic rating model, the model of paired comparisons, maximum likelihood inference, existence and uniqueness of maximum likelihood estimates, Bayesian inference, factor graphs, approximate assumed density filtering, Gibbs sampling, rating systems Elo, Glicko, TopCoder, and TrueSkill.
9. Information labeling: simple majority decoding, weighted majority decoding, optimality of weighted majority rule, optimal assignment of labeling tasks to workers.

Pre-requisites

Familiarity with the basic concepts of game theory will be useful but not assumed, e.g. those covered in the course Mathematics of Operational Research MMath/MASt (Part III).

Literature

1. M. Vojnović, Contest Theory, Cambridge University Press, book manuscript in preparation (2014).
2. V. Krishna, Auction Theory, Academic Press (2002).
3. E. Law and L. von Ahn, Human Computation, Morgan & Claypool (2011).
4. N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, Algorithmic Game Theory, Cambridge University Press (2007).
5. M. Osborne and A. Rubenstein, A Course in Game Theory, MIT Press (1994).

Additional support

Two examples sheets will be provided and two associated examples classes will be given.

Particle Physics, Quantum Fields and Strings

The courses on Symmetries, Fields and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of the course Symmetries, Fields and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year		*Courses*
Second	*Essential*:	Quantum Mechanics, Methods, Complex Methods.
	Helpful:	Electromagnetism.
Third	*Essential*:	Principles of Quantum Mechanics, Classical Dynamics.
	Very helpful:	Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Field Theory (M24)

Malcolm J. Perry

Quantum Field Theory is the language in which all of modern physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics. How these feilds interact with a classical electromagnertic field is described.

Next, we introduce the path integral which is an alternative way of describing quantum fields. The paht integral is fundamental in introducing interaction into quantum field theory. Interactions are described using perturbative theory and Feynman diagrams. This is first illustrated for theories with a purely scalar field interaction, and then for a couplings between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processes are computed.

Finally, the idea of loops in Feynman diagrams are explored and the queston of the consequent infinities looked at. Ways of dealing with the infinities will be expolored in the Advanced Quantum Field Theory course which follows on directly from this one.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Literature

1. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).
2. A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2010)
3. M. Srednicki, *Quantum Field Theory*, Cambridge University Press, (2007). (a free prelminary version is available here:http://web.physics.ucsb.edu/ mark/ms-qft-DRAFT.pdf)
4. M. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (2014).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Symmetries, Fields and Particles (M24)

N.S. Manton

This course introduces the various types of elementary particle – quarks, leptons, gauge and Higgs particles – and the various symmetry groups useful for understanding their properties. Some symmetries in particle physics are exact, and some only approximate. The most important symmetry groups are specific Lie groups, including SU(2), SU(3), the Lorentz and Poincaré groups.

Basic Lie group theory will be covered, including Lie algebras and the relation between Lie algebras and Lie groups. The representation theory of SU(2) (closely related to quantum mechanical angular momentum theory) will be extended to give the theory of SU(3) representations. Hadrons, the particles containing quarks and antiquarks, are classified by representations of SU(3) because of the approximate flavour symmetry among quarks.

The Standard Model of particles is a gauge theory, a quantum field theory with an exact, locally acting Lie group symmetry. The structure of gauge theory Lagrangians will be introduced, and also the Higgs mechanism for (spontaneous) symmetry breaking and mass generation.

The course ends with a discussion of the Lorentz and Poincaré groups, and how their representations are used to classify momentum and spin states of relativistic particles.

The course is designed to be taken in conjunction with the Quantum Field Theory course, and as a preliminary to the Standard Model course, although it is formally independent of these.

Pre-requisites

Basic finite group theory, including subgroups and orbits. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

It will be useful to have an outline knowledge of elementary particles, as in several books, e.g. D.H. Perkins, Introduction to High Energy Physics, 4th ed., Cambridge University Press, 2000.

Literature

1. G. Costa and G. Fogli, *Symmetries and Group Theory in Particle Physics, Lecture Notes in Physics 823.* Springer, 2012.
2. H.F. Jones, *Representations and Physics.* 2nd edition. Taylor and Francis, 1998.
3. H. Georgi, *Lie Algebras in Particle Physics.* Westview Press, 1999.
4. J. Fuchs and C. Schweigert, *Lie Algebras and Representations.* Cambridge University Press, 2003.

Additional support

A set of course notes is available on the Part III Examples and Lecture Notes webpage. Four examples sheets will be provided and four associated examples classes in moderate-sized groups will be given by graduate students.

Statistical Field Theory (M16)

R R Horgan

This course is an introduction to the renormalization group, the basis for a modern understanding of field theory, and the construction of effective field theories. The discussion is concerned with statistical systems including their relationship with quantum field theory in its Euclidean formulation.

The phenomenology of phase transitions is reviewed, leading to the introduction of the theory of critical phenomena. Landau-Ginsburg theory and mean field theory are presented and applied to the Ising model. The classification of phase transitions and their relationship with critical points is presented, and the renormalization group is introduced first in the context of the soluble 1D Ising model and then in general. The renormalization group is used for calculating properties of systems near a phase transition, for example in the Ising and Gaussian models, and the concepts of critical exponents, anomalous dimensions, and scaling are discussed.

The idea of the continuum limit for models controlled by a critical point and the relationship with continuum quantum field theory is elucidated.

Perturbation theory is introduced for the scalar field model with interactions and some example calculations are presented.

Books

1. J M Yeomans, *Statistical Mechanics of Phase Transitions*, Oxford Scientific Publishing (1992)
2. L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press (1996)
3. J Cardy, *Scaling and Renormalization in Statistical Physics*, Cambridge Lecture Notes in Physics (1996)

4. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992)

5. D Amit and V Martín-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005)

6. C Itzykson and J-M Drouffe, *Statistical Field Theory*, Vols. 1-2, Cambridge University Press (1991)

Pre-requisites

background knowledge of Statistical Mechanics at an undergraduate level is essential. Although not a formal prerequisite, attendance at the Part III course Quantum Field Theory is a considerable advantage and strongly advised.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Analysis of Gauge Theories (M16)

Non-Examinable (Graduate Level)

David Stuart

An introduction to analysis of gauge theories will be given, including both variational and PDE methods. As time allows we will cover aspects of: Gauge fixing. Compactness. Existence theorems for static solutions. Yang-Mills gradient flow, Yang-Mills equations on Minkowski space and related systems of equations.

Pre-requisites

Undergraduate analysis. Basic Integration and Measure theory, PDE and Calculus of Variations, or a willingness to learn concurrently as necessary.

Literature

1. Jaffe, A. and Taubes, C. *Vortices and Monopoles: Structure of Static Gauge Theories.* Birkhauser, Boston (1980).

2. Taubes, C. *Differential geometry. Bundles, connections, metrics and curvature.* Oxford Graduate Texts in Mathematics, 23 (2011).

3. Uhlenbeck, Karen K. Connections with Lp bounds on curvature. *Comm. Math. Phys.* 83 (1982), no. 1, 3142.

4. Uhlenbeck, Karen K. Removable singularities in Yang-Mills fields. *Comm. Math. Phys.* 83 (1982), no. 1, 1129.

Advanced Quantum Field Theory (L24)

DB Skinner

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalisation of electrodynamics and form the backbone of the Standard Model - our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantising a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson Loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study Renormalization. Wilsons picture of Renormalisation is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the Renormalisation Group (RG) flow. The course explores renormalisation systematically, from the use of dimensional regularisation in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as asymptotic freedom, this phenomenon revolutionised our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrise possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Time permitting, we may also discuss various modern topics in QFT, such as dualities, localization and topological QFTs,

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

Preliminary Reading

1. Zee, A., *Quantum Field Theory in a Nutshell*, 2$^{\text{nd}}$ edition, PUP (2010).

Literature

1. Srednicki, M., *Quantum Field Theory*, CUP (2007).
2. Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).
3. Banks, T. *Modern Quantum Field Theory: A Concise Introduction*, CUP (2008).
4. Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).

Additional support

There will be four problem sheets handed out during the course. Classes for each of these sheets will be arranged during Lent Term. There will also be a general revision class during Easter Term.

Standard Model (L24)

M.B. Wingate

The Standard Model of particle physics is, by far, the most successful application of quantum field theory (QFT). As this booklet goes to press, this model accurately describes all experimental measurements involving strong, weak, and electromagnetic interactions.

The Standard Model is the quantum theory of the gauge group $SU(3) \times SU(2) \times U(1)$ with fermion fields for the leptons and quarks. The course aims to demonstrate how this model is realised in nature. It is intended to complement the more general Advanced QFT course.

This course begins by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content in terms of spin 1/2 leptons and quarks and also the spin 1 gauge bosons. The parity P, charge conjugation C and time-reversal T transformation properties of the theory are investigated. These need not be symmetries manifest in nature; e.g. only left-handed particles feel the weak force in violation of parity symmetry. We show how CP violation becomes possible when there are three generations of particles.

Ideas of spontaneous symmetry breaking are applied to discuss the Higgs Mechanism; the weakness of the weak force is due to the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry. The recent measurements of what appear to be Higgs boson decays will be presented.

We show how to obtain cross-sections and decay rates from the matrix element squared of a process. Various scattering and decay processes can be calculated in the electroweak sector using perturbation theory because of the smallness of the couplings. We touch upon the topic of neutrino masses and oscillations, an important window into physics beyond the Standard Model.

The strong interactions are based upon the gauge theory with (unbroken) gauge group $SU(3)$, called quantum chromodynamics (QCD). At low energies quarks are confined, forming bound states called hadrons. In such a non-abelian theory, the coupling constant decreases in higher energy processes to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. Nonperturbatively, progress can be made in the limits of very small and very large quark masses, making use of chiral and heavy quark symmetries. We introduce the framework of effective field theory and apply it to QCD.

Very high energy experiments and very precise experiments are currently striving to observe effects not describable by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model.

Four examples sheets and classes complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and Symmetries of Particles and Fields courses, or to be familiar with the material covered in them. It is advantageous to attend the Advanced QFT course during the same term as attending this course, or to study renormalisation and non-abelian gauge fixing.

Reading to complement course material

1. M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley (1996).
2. H. Georgi, Weak Interactions, Benjamin/Cummings (1984).
3. T-P. Cheng and L-F. Li, Gauge Theory of Elementary Particle Physics, Oxford University Press (1984).
4. I.J.R. Aitchison and A.J.G. Hey, Gauge Theories in Particle Physics, IoP Publishing (1989).

5. F. Halzen and A.D. Martin, Quarks and Leptons: An Introductory Course in Modern Particle Physics, John Wiley and Sons (1984).

6. J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model, Cambridge University Press (1994).

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Supersymmetry and Extra Dimensions (L24)

Ben Allanach and Fernando Quevedo

This 24 lecture course provides an introduction to the use of supersymmetry (roughly 2/3) and extra dimensions (roughly 1/3) in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables and relates fermions and bosons.

Firstly, a physical motivation for supersymmetry is provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced.

Extra spatial dimensions are introduced in Kaluza Klein theories and gravitation is discussed. Brane worlds and warped compactification are introduced and then supersymmetry in higher dimensions is presented.

Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them.

For more advanced topics later in the course, it will helpful but not essential to have a knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model in order to understand the lecture on the minimal supersymmetric standard model.

Literature

For a gentle introduction of the subject, see

1. The first chapters of `http://arxiv.org/abs/hep-ph/0505105`

Beware: most of the supersymmetry references contain errors in minus signs, aside (as far as we know) Wess and Bagger.

1. Course lecture notes from last year: `http://www.damtp.cam.ac.uk/user/examples/3P7.pdf`

2. Videos of a very similar lecture course: follow the links from
`http://users.hepforge.org/˜allanach/teaching.html`

3. Supersymmetric Gauge Field Theory and String Theory, Bailin and Love, IoP Publishing (1994) has nice explanations of the physics. An erratum can be found at
`http://www.phys.susx.ac.uk/m̃pfg9/susyerta.htm`

4. Introduction to supersymmetry, J.D. Lykken, `hep-th/9612114`. This introduction is good for extended supersymmetry and more formal aspects.

5. Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this terse and more mathematical book has the opposite sign of metric to the course.

6. A supersymmetry primer, S.P. Martin, `hep-ph/9709256` is good and detailed for phenomenological aspects, although with the opposite sign metric to the course.

7. On supersymmetry and extra dimensions part of Weinberg's Field Theory (Vol 3) book, Cambridge University Press.

8. See also Introduction to Supergravity by H. Nastase, arXiv:1112.3502

9. On extra dimensions TASI lectures by R. Sundrum hep-th/0508134 and Cargese lectures by R. Rattazzi arXiv:hep-ph/0607055.

Additional support

Four examples sheets will be provided and three associated examples classes will be given.

String Theory (L24)

Paul Townsend

The basic idea of String Theory is that elementary particles are excitations of a relativistic string, which could be open (i.e. with two endpoints) or closed. Each quantum excitation of the string behaves like an elementary particle, and closed strings have a massless spin-2 particle in their spectrum, which suggests that String Theory is a theory of quantum gravity. Open strings yield analogous generalisations of gauge theory, so a theory of open and closed strings is potentially one that can unify gravity with the forces of the standard model of particle physics.

This course will introduce the strings of string theory as examples of constrained Hamiltonian dynamics. Various methods of quantisation (light-cone, covariant, path-integral) will be illustrated using the point particle and applied to the free Nambu-Goto (bosonic) string. This will reveal that there is a critical space-time dimension (26 for the Nambu-Goto string) and that the ground state is a tachyon. A study of the possible boundary conditions on open strings will reveal that String Theory also includes "branes". The "spinning string" , with critical space-time dimension 10, will be introduced as a generalisation of the "spinning particle". How this leads to tachyon-free superstring theories will be briefly discussed, with an even briefer discussion of why there are five of them.

Interactions of strings will be introduced using Euclidean path-integral methods, with a brief introduction to conformal field theory. The simplest scattering amplitudes for particles in the the spectrum of the Nambu-Goto string (Veneziano, Virasoro) will be derived and analysed. General principles of perturbation theory that follow from the path-integral approach will be used to introduce the idea of an effective spacetime action as a double expansion in the inverse string tension and a string coupling constant, reducing to 10-dimensional GR (or supergravity) at leading order. This will provide the basis for a brief discussion of how the five superstring theories are unified by an 11-dimensional "M-Theory".

Pre-requisites

This course assumes only a basic knowledge of classical and quantum mechanics, although some previous (or parallel) exposure to QFT and GR will be useful.

Literature

In this course, String Theory is developed from a Hamiltonian perspective that does not have much overlap with current texts. However, the books that are likely to be most useful are

1. Green, Schwarz and Witten, *Superstring Theory: Vol. 1*, CUP 1987.

2. Lüst and Theisen, *Lecture Notes in Physics: Superstring Theory.* Springer, 1989.

3. Brink and Henneaux, *Principles Of String Theory.* Plenum, 1988.

Additional support

Complete course notes will be provided, along with four examples sheets.

Classical and Quantum Solitons. (E16)

N. Dorey

Solitons are solutions of the classical field equations with particle-like properties. In particular, they are localised in space, have finite energy and are stable against decay into radiation. After quantisation, they give rise to new particle states which are typically very massive at weak coupling but can become light at strong coupling. Solitons play a key role in many recent advances in field theory and string theory, especially in the phenomenon of duality which relates the strong-coupling behaviour of one theory to the weak-coupling behaviour of another. In this course we will study the properties of classical solitons and their quantum counterparts. We will focus mainly on the case of integrable theories in two dimensional spacetime where an exact analytic description is possible.

Pre-requisites

Quantum Field Theory. Advanced Quantum Field Theory.

Literature

1. Topological Solitons, N. Manton and P. Sutcliffe (CUP 2004), Chapters 1, 4 and 5

Additional support

Two examples sheets will be provided and two associated examples classes will be given.

Introduction to the Gauge/Gravity duality (E16)

J. E. Santos

The Gauge/Gravity duality relates properties of $d-$dimensional gravity theories with those of certain $(d-1)-$dimensional quantum theories where gravity is absent. Moreover, in most known examples, the duality is of the *strong-weak* type: whenever the gravitational theory is classical and well described by a two derivative action, the field theory has a large number of degrees of freedom and is strongly coupled. The converse is also true, that is, when the gravitational theory is highly quantum, the field theory is perturbative. The importance of these dualities stems from both our rudimentary understanding of generic strongly coupled field theories, and from our poor understanding of quantum gravity.

The dictionary between the two sides of the duality is best known in cases where the field theory is conformal, and has an underlying supersymmetric UV completion. The primary example being the duality between four-dimensional $\mathcal{N} = 4$ Super-Yang-Mills with gauge group $SU(N)$ and IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. This correspondence has been the subject of many nontrivial checks, but a mathematical proof remains an outstanding problem.

In these lectures we aim to discuss the following:

1. Basics of conformal field theory.

2. General relativity in anti-de-Sitter space.

3. The correspondence: its formulation and dictionary.
4. Linear response theory in quantum field theory.
5. Transport properties in holography.
6. The fluid-gravity correspondence.

Pre-requisites:

Knowledge of the Michaelmas and Lent terms courses General Relativity, Quantum Field Theory and Advanced Quantum field theory will be assumed. Familiarity with the contents of the Lent course Black holes is highly recommended.

Introductory reading:

1. J. M. Maldacena, Chapter 12 of *Black holes in higher dimensions*, edited by G. T. Horowitz, 2012, Cambridge University Press.

Reading to complement course material:

1. J. McGreevy *Holographic duality with a view toward many-body physics*, arXiv:0909.0518 [hep-th].
2. O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, *Large N Field Theories, String Theory and Gravity*, hep-th/990511.
3. J. M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, hep-th/9711200 - **The original paper**.
4. S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, hep-th/9802109.
5. E. Witten, *Anti De Sitter Space And Holography*, hep-th/9802150.

Additional support:

Three examples sheets will be provided and three associated examples classes will be given.

Relativity and Gravitation

These courses provide a thorough introduction to General Relativity and Cosmology. The Michaelmas term courses introduce these subjects, which are then developed in more detail in the Lent term courses on Black Holes and Advanced Cosmology. Applications of Differential Geometry to Physics explains how many physical theories can be formulated elegantly using the language of differential geometry. Non-examinable courses explore more advanced topics.

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year		*Courses*
First	*Essential*:	Vectors & Matrices, Diff. Eq., Vector Calculus, Dynamics & Relativity.
Second	*Essential*:	Methods, Quantum Mechanics, Variational Principles.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Essential*:	Classical Dynamics.
	Very helpful:	General Relativity, Stat. Phys., Electrodynamics, Cosmology.
	Helpful:	Further Complex Methods, Asymptotic methods.

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation.

General Relativity (M24)

Ulrich Sperhake

An axiomatic development of the subject will be given.

Pre-requisites

This course will be self-contained, so previous knowledge of General Relativity is not essential. However, many students have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before, then it is strongly recommended that you study an introductory book (e.g. Hartle or Schutz) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Familiarity with Newtonian Gravity and special relativity is essential. Knowledge of the relativistic formulation of electrodynamics is desirable. Familiarity with finite-dimensional vector spaces, the calculus of functions $f : R^m \to R^n$, and the Euler-Lagrange equations will be assumed.

Literature

Introductory Reading

1. J. B. Hartle, *An introduction to Einstein's General Relativity.* Addison-Wesley, 2003.
2. B. Schutz, *A First Course in General Relativity.* Cambridge University Press, 2009.

Reading to complement course material

1. R. M. Wald, *General Relativity.* University of Chicago Press, 1984.
2. S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity.* Addison-Wesley, 2004.
3. J. M. Stewart, *Advanced General Relativity.* Cambridge University Press, 1993.
4. L. Ryder, *Introduction to General Relativity.* Cambridge University Press, 2009.
5. E. Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity.*

   `http://arxiv.org/abs/gr-qc/0703035` .

Chapter 1 of John Stewart's book gives a concise overview of differential geometry which also guides this part of the course. Carroll's and Ryder's books are very readable introductions. Gourgoulhon's notes provide a comprehensive overview of the space-time split of general relativity. Wald's book discusses many advanced topics; very suitable for obtaining comprehensive treatment on isolated topics.

Additional support

Three examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Course website:

`http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html`

Cosmology (M24)

Daniel Baumann

This course is about 13.8 billion years of cosmic evolution, from primordial quantum fluctuations to the formation of galaxies.

A tentative syllabus is the following:

1. Geometry and Dynamics
2. Inflation
3. Thermal History
4. Cosmological Perturbation Theory
5. Initial Conditions from Inflation
6. Structure Formation
7. Cosmic Microwave Background

Further details may be found on the course website:

`http://www.damtp.cam.ac.uk/user/db275/Cosmology.pdf`

Pre-requisites

Basic knowledge of relativity, quantum mechanics and statistical mechanics will be helpful. However, the course will be presented in a self-contained way, so students with less experience in any of these fields should have no problem to catch up.

Literature

Introductory reading

1. Weinberg, *The First Three Minutes.*
2. Carroll, *A No-Nonsense Introduction to General Relativity.*

Reading to complement course material

1. Dodelson, *Modern Cosmology.*
2. Kolb and Turner, *The Early Universe.*
3. Weinberg, *Cosmology.*
4. Mukhanov, *Physical Foundations of Cosmology.*
5. Peter and Uzan, *Primordial Cosmology.*

Additional support

Detailed lecture notes will be made available on the course website. Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Numerical General Relativity (L16)

Non-Examinable (Graduate Level)

Dr. P. Figueras and Dr. H. Witek

General Relativity (GR) is our most succesful theory of gravity to date and finds applications in astrophysics and cosmology as well as high energy physics. There are many interesting problems including finding new types of stationary black hole solutions and investigating highly dynamical systems, such as the 2-body problem in GR, which cannot be solved analytically and instead require a numerical treatment.

The first part of this course will be devoted to a general introduction to the PDEs that more often arise in mathematical physics as well as the numerical methods to solve them. We will introduce finite difference methods, spectral methods and (time permitting) finite element methods.

In the second part of the course we will discuss how to solve the Einstein equations for stationary (i.e., time independent) problems. We will first discuss how one can formulate the problem in a way such that the equations are manifestly elliptic and can, therefore, be solved as a standard boundary value problem. We will introduce the general techniques to solve such problems, namely Ricci flow and Newton's method, through practical examples.

The third part of this course will focus on the time evolution problem in GR, which allows to explore highly dynamical systems in the strong curvature regime. We will derive the formulation of Einstein's equations as Cauchy problem and describe techniques to solve for the initial data and its evolution in time. We will discuss the numerical stability of evolution schemes which is closely connected to the well-posedness of the underlying PDE system. We will further present the BSSN and Generalized Harmonic formulations of the initial value problem which are widely used throughout the Numerical Relativity community. If time permits, we will discuss extensions of the "standard" Numerical Relativity techniques in 4-dimensional, asymptotically flat spacetimes to higher dimensions and more generic asymptotics.

Pre-requisites

Pre-requisites include knowledge of General Relativity (at undergraduate level) and the theory of partial differential equations. Programming skills and familiarity with computer algebra programs such as mathematica are not mandatory but would be useful.

Literature

The following texts will cover the majority of the course, and are available online.

1. A. Iserles, *A first course in the numerical analysis of differential equations*, Cambridge Texts in Applied Mathematics, CUP, 2008.
2. Ll. Trefethen, *Spectral Methods in Matlab*, SIAM, 2000.
3. T. Wiseman, *Numerical construction of static and stationary black holes*, arXiv:1107.5513 [gr-qc].
4. M. Alcubierre, *Introduction to 3+1 numerical relativity* , International series of monographs on physics, Oxford Univ. Press, Oxford, 2008;
5. T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity*, Cambridge University Press, 2010; see also `http://arxiv.org/abs/gr-qc/0211028`
6. E. Gourgoulhon *3+1 formalism and bases of numerical relativity*, 2007,

   `http://arxiv.org/abs/gr-qc/0703035`
7. H. Witek, *Lecture Notes: Numerical Relativity in higher dimensional spacetimes*, International Journal of Modern Physics A Vol. **28**, 1340017 (2013); see also `http://arxiv.org/abs/arXiv:1308.1686`
8. D. Hilditch, *An Introduction to Well-posedness and Free-evolution*, International Journal of Modern Physics A Vol. **28**, 1340015 (2013); see also `http://arxiv.org/abs/arXiv:1309.2012`

Black Holes (L24)

Harvey Reall

A black hole is a region of space-time that is causally disconnected from the rest of the Universe. The study of black holes reveals many surprising and beautiful properties, and has profound consequences for quantum theory. The following topics will be discussed:

1. Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
2. The initial value problem, strong cosmic censorship.
3. Causal structure, null geodesic congruences, Penrose singularity theorem.
4. Penrose diagrams, asymptotic flatness, weak cosmic censorship.
5. Reissner-Nordstrom and Kerr black holes.
6. Energy, angular momentum and charge in curved spacetime.
7. The laws of black hole mechanics. The analogy with laws of thermodynamics.
8. Quantum field theory in curved spacetime. The Hawking effect and its implications.

Pre-requisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

1. H. S. Reall, *Part 3 Black Holes*: lecture notes available at

   `http://www.damtp.cam.ac.uk/user/hsr1000`

2. R.M. Wald, *General relativity*, University of Chicago Press, 1984.
3. S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
4. V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.
5. N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. R.M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be distributed during the course. Four examples classes will be held to discuss these.

Advanced Cosmology (L24)

Paul Shellard and Anthony Challinor

This course will take forward at much greater depth the topics in modern cosmology covered at the end of the Michaelmas Term course. The prediction from fundamental theory for the statistics properties of the primordial perturbations remains the key area of confrontation with cosmological observations, both from large-scale structure and the cosmic microwave background (CMB). This course will develop the mathematical tools and physical understanding necessary for research in this very active area.

Cosmological inhomogeneity theory

- The $3 + 1$ spacetime formalism and the Einstein equations
- General linearised perturbation equations in an expanding universe
- Quadratic and cubic action for nonlinear perturbations.
- Statistics of random fields

Cosmic microwave background

- Relativistic kinetic theory
- The Boltzmann equation
- Photon scattering and diffusion
- The CMB temperature power spectrum
- Primordial gravitational waves and the CMB
- CMB Polarization

Topical issues: non-Gaussianity from inflation

- Higher-order correlation functions
- "In-in" formalism and non-Gaussianities from ination
- CMB non-Gaussianity and other observational prospects

Pre-requisites

Material from the Michaelmas term *Cosmology* is essential. Familiarity with introductory Quantum Field Theory is recommended.

Literature

Textbooks

1. Dodelson, S., *Modern Cosmology*, Elsevier (2003).
2. Mukhanov, V., *Physical Foundation of Cosmology*, Cambridge (2005).
3. Weinberg, S., *Cosmology*, Oxford University Press (2008).
4. Misner, C.W., Thorne, K.S., and Wheeler, J.A., *Gravitation*, Freeman (1973).
5. Durrer, R., *The Cosmic Microwave Background*, Cambridge (2008).

Useful references

1. Bardeen, J.M., *Cosmological Perturbations From Quantum Fluctuations To Large Scale Structure*, DOE/ER/40423-01-C8 Lectures given at 2nd Guo Shou-jing Summer School on Particle Physics and Cosmology, Nanjing, China, Jul 1988. (Available on request.)
2. Mukhanov, V.F., Feldman, H.A., and Brandenberger, R.H., *Theory of cosmological perturbations*, Physics Reports, 215, 203 (1992).
3. Ma, C., and Bertschinger, E., *Cosmological Perturbation Theory in Synchronous and Conformal Newtonian Gauges*, Astrophysical Journal, 455, 7 (1995) [astro-ph/9506072].
4. Hu, W. and White, M., *CMB anisotropies: Total angular momentum method*, Physical Review D, 56, 596 (1997) [astro-ph/9702170].
5. Hu, W. and White, M., *A CMB polarization primer*, New Astronomy, 2, 323 (1997) [astro-ph/97006147].
6. Maldacena, J., *Non-gaussian features of primordial fluctuations in single field inflationary models*, Journal of High Energy Physics, 5, 13 (2003).
7. Chen, X., *Primordial Non-Gaussianities from Ination Models* [arxiv:1002.1416].
8. Ligouri, M., Sefusatti, E., Fergusson, J.R., and Shellard, E.P.S., *Primordial Non-Gaussianity and Bispectrum Measurements in the Cosmic Microwave Background and Large-Scale Structure*, Advances in Astronomy, 2010, 73 (2010) [arxiv:1001.4707]

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Applications of Differential Geometry to Physics. (L16)

Maciej Dunajski

This is a course designed to develop the Differential Geometry required to follow modern developments in Theoretical Physics. The following topics will be discussed.

- Differential Forms and Vector Fields.
  1. One parameter groups of transformations.
  2. Vector fields and Lie brackets.
  3. Exterior algebra.
  4. Hodge Duality.
- Geometry of Lie Groups.
  1. Group actions on manifolds.
  2. Homogeneous spaces and Kaluza Klein theories.
  3. Metrics on Lie Groups.
- Fibre bundles and instantons.
  1. Principal bundles and vector bundles.
  2. Connection and Curvature.
  3. Twistor space.

Pre-requisites

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

Literature

1. `http://www.damtp.cam.ac.uk/research/gr/members/gibbons/gwgPartIII_DGeometry2011-1.pdf`
2. Flanders, H. Differential Forms. Dover
3. Dubrovin, B., Novikov, S. and Fomenko, A. Modern Geometry. Springer
4. Eguchi, T., Gilkey, P. and Hanson. A. J. Physics Reports 66 (1980) 213-393
5. Arnold. V. Mathematical Methods of Classical Mechanics. Springer.
6. Dunajski. M. Solitons, Instantons and Twistors. OUP.

Additional support

Two examples sheets will be provided and two associated examples classes will be given.

Spinor Techniques in General Relativity (L24)

Non-Examinable (Graduate Level)

Irena Borzym (12 Lectures) and Peter O'Donnell (12 Lectures)

Spinor structures and techniques are an essential part of modern mathematical physics. This course provides a gentle introduction to spinor methods which are illustrated with reference to a simple 2-spinor formalism in four dimensions. Apart from their role in the description of fermions, spinors also often provide useful geometric insights and consequent algebraic simplifications of some calculations which are cumbersome in terms of spacetime tensors.

The first half of the course will include an introduction to spinors illustrated by 2-spinors. Topics covered will include the conformal group on Minkowski space and a discussion of conformal compactifications, geometry of scri, other simple simple geometric applications of spinor techniques, zero rest mass field equations, Petrov classification, the Plucker embedding and a comparison with Euclidean spacetime. More specific references will be provided during the course and there will be worked examples and handouts provided during the lectures.

The second half of the course will include: Newman-Penrose (NP) spin coefficient formalism, NP field equations, NP quantities under Lorentz transformations, Geroch-Held-Penrose (GHP) formalism, modified GHP formalism, Goldberg-Sachs theorem, Lanczos potential theory, Introduction to twistors. There will be no problem sets.

Pre-requisites

The Part 3 general relativity course is a prerequisite.

No proir knowledge of spinors will be assumed.

Literature

Introductory material.

1. L. P. Hughston and K. P. Tod, *Introduction to General Relativity.* Freeman, 1990.
2. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation.* Freeman, 1973.

Best Course Reference Text for Lectures 1 to 12.

J.M. Stewart, *Advanced General Relativity.* CUP, 1993.

Best Course Reference Text for Lectures 13 to 24.

P ODonnell, *Introduction to 2-spinors in general relativity.* World Scientific, 2003.

Reading to complement course material.

1. Penrose and Rindler, *Spinors and Spacetime Volume 1.* Cambridge Monographs on Mathematical Physics, 1987.
2. S. Ward and Raymond O. Wells, *Twistor Geometry and Field theory.* Cambridge Monographs on Mathematical Physics, 1991 .
3. Robert J. Baston, Michael G. Eastwood, *The Penrose Transform.* Clarendon Press, 1989.
4. S. A Huggett and P. Tod, *Introduction to Twistor Theory.* World Scientific, 2003.
5. R.M. Wald, *General Relativity.* World Chicago UP, 1984.
6. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime.* CUP, 1973.

Astrophysics

Introduction to Astrophysics courses

These courses provide a broad introduction to research in theoretical astrophysics; they are taken by students of both Part III Mathematics and Part III Astrophysics. The courses are mostly self-contained, building on knowledge that is common to undergraduate programmes in theoretical physics and applied mathematics. For specific pre-requisites please see the individual course descriptions.

Astrophysical Fluid Dynamics (M24)

Gordon Ogilvie

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as determining their stability and their response to tidal forcing.

The aim of the course is to provide familiarity with the basic phenomena and techniques that are of general relevance to astrophysics. Wherever possible the emphasis will be on simple examples, physical interpretation and application of the results in astrophysical contexts.

Provisional synopsis

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Stellar oscillations. Introduction to asteroseismology and astrophysical tides.
- Local dispersion relation. Internal waves and instabilities in stratified rotating astrophysical bodies.

Pre-requisites

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas.* Cambridge University Press.
2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.
3. Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows.* Cambridge University Press. (Available as an e-book from `http://ebooks.cambridge.org`)
4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics.* University Science Books.
5. Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics.* Imperial College Press.

Additional support

Four example sheets will be provided and four associated classes will be given by the lecturer. It is anticipated that extended notes supporting the lecture course will be available in electronic form.

Structure and Evolution of Stars (M24)

A.N.Żytkow

Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a mathematical description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These basic equations have to be supplemented by a number of appropriately chosen equations describing the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed. Some familiarity with the principles of hydrodynamics, thermodynamics, quantum mechanics, atomic and nuclear physics will be assumed.

Approximate solutions of the equations will be shown; polytropic gas spheres, homology principles, the virial theorem will be presented. The evolution of a star will be discussed, starting from the main-sequence, following the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

The only way in which we may test stellar structure and evolution theory is through comparison of the theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

Desirable Previous Knowledge

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics although a detailed knowledge of all of these is not expected.

Introductory Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Reading to complement course material

1. Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution, Second Edition*, Springer-Verlag, 2012.
2. Iben, I. *Stellar Evolution Physics, Vol. 1 and 2.* Cambridge University Press, 2013.
3. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
4. Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

Additional support

There will be four example sheets each of which will be discussed during an examples class. There will be a one-hour revision class in the Easter Term.

Extrasolar Planets - Atmospheres and Interiors (M24)

Nikku Madhusudhan

The field of extrasolar planets (or 'exoplanets') is one of the major frontiers of modern astronomy. Over a thousand exoplanets are now known with a wide range of sizes, temperatures, and orbital parameters, covering all the categories of planets in the solar system (gas giants, ice giants, and rocky planets) and more. The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and stratospheres, equilibrium/non-equilibrium chemistry, atmospheric dynamics, clouds/hazes, etc) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, as well as parametric models and retrieval techniques) (3) models of exoplanetary interiors (Equations of state, mass-radius relations, interiors of giant planets vs super-Earths, water worlds, diamond planets, etc.), and (4) relating atmospheres and interiors to planet formation. Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres of all kinds. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed. The course will also include a discussion on detecting biosignatures in rocky exoplanets, the relevant theoretical constructs and expected observational prospects with future facilities.

Pre-requisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self contained and basic concepts will be introduced for completeness.

Literature

1. Chapters on exoplanetary atmospheres and interiors in the book *Protostars and Planets VI*, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning. Most of these chapters are available publicly on the astro-ph arXiv.
2. Seager, S., *Exoplanet Atmospheres: Physical Processes*, Princeton Series in Astrophysics (2010).
3. Seager, S. *Exoplanets*, University of Arizona Press (2011), ed. S. Seager.
4. de Pater, I. and Lissauer J., *Planetary Sciences*, Cambridge University Press (2010).

Additional support

3-4 examples sheets will be provided and 3-4 associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Magnetohydrodynamics (M16)

Prof. M.R.E.Proctor

Magnetohydrodynamics is the study of the interaction between magnetic fields and conducting fluids. Two main effects are of interest. Firstly moving conducting fluid can generate electric currents from magnetic fields through Faraday induction, and this leads to changes in the magnetic field. For sufficiently vigorous flows magnetic fields can be self-excited ('fluid dynamos'), and this process is responsible for the generation of magnetic fields in the Earth, Sun and other astrophysical bodies. Quite recently fluid dynamos have been demonstrated in the laboratory. While induction is a linear process, nonlinearity is induced since magnetic fields exert forces on the fluid, and these are proportional to the square of the field strength. This interaction leads to new types of wave motion ('Alfven waves') in conducting magnetised fluids, and has large scale effects of for example the statistics of fully developed turbulence and the morphology of sunspots. The course will treat both the basic theory and a number of applications as time permits. The theory will be developed in a classical rather than relativistic framework.

Pre-requisites

Knowledge of fluid dynamics and basic electrodynamics would be an advantage.

Literature

1. Moffatt, H.K. Generation of magnetic fields in conducting fluids. C.U.P. (out of print)
2. Priest, E.R. Solar magnetohyrodynamics. Kluwer.
3. Dormy and Soward, eds. Mathematical Aspects of Natural Dynamos. CRC Press
4. S.Chandrasekhar. Hydrodynamics and Hydromagnetic Stability. Dover.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The Origin and Evolution of Galaxies (M16)

Martin Haehnelt

Galaxies are a fundamental building block of our Universe. The course will give an account of the physics of the formation of galaxies and their central supermassive black holes in the context of the standard paradigm for the formation of structure in the Universe.

Specific topics to be covered include the following:

- Observed properties of galaxies
- Cosmological framework and basic physical processes
- The interplay of galaxies and the intergalactic medium from which they form
- Numerical Methods for modeling galaxy formation

- Collapse of dark matter haloes and the inflow/outflow of baryons
- The hierarchical build-up of galaxies
- The origin and evolution of the central supermassive black holes in galaxies
- Towards understanding the origin of the Hubble sequence of galaxies

Pre-requisites

The course is aimed at astronomers/astrophysicists (including beginning graduate students). It should be also suitable for interested physicists and applied mathematicians. The course is self-contained, but students who have previously attended introductory courses in General Relativity and/or Cosmology will have an easier start.

Literature

1. Mo, H., van den Bosch, F., White, S., *Galaxy Formation and Evolution*, 2010, Cambridge University Press.
2. Sparke, L., Gallagher, J.S., *Galaxies in the Universe*, 2nd ed., 2007, Cambridge University Press.
3. Schneider, P., *Extragalactic Astronomy and Cosmology: An Introduction*, 2006, Springer.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Galactic Dynamics (L24)

Wyn Evans

Astrophysics provides many examples of complex dynamical systems. This course covers the mathematical tools to describe Galaxies as well as reviewing their observational properties. The behaviour of these systems is controlled by Newton's laws of motion and Newton's law of gravity. Galaxies are dynamically very young, a typical star like the Sun having orbited only thirty or so times around the galaxy. The motions of stars in Galaxies are described using classical statistical mechanics, since the number of stars is so great. The study of large assemblies of stars interacting via long-range forces provides many unusual examples of cooperative phenomena, such as bars and spiral structure. The interplay between astrophysical dynamics and modern cosmology is also important – much of the evidence for dark matter is dynamical in origin.

1. Observational overview. Stellar populations in galaxies, galaxy morphology and classification. Dust and gas in galaxies. Scaling Laws.
2. Theory of the gravitational potential. Poisson's equation. Spherical, spheroidal and disk-like systems.
3. Regular and chaotic orbits, the epicyclic approximation, surfaces of section, integrals of motion, action-angle coordinates, adiabatic invariance.
4. Collisionless stellar dynamics, the Boltzmann equation, the Jeans Theorem, the Jeans equations, equilibrium models, astrophysical applications.
5. Collisional dynamics, the Fokker-Planck equation, dynamical friction.
6. Globular cluster evolution, evaporation and ejection, the gravothermal catastrophe, the effect of hard and soft binaries.
7. Galactic stability, the Jeans length, theories of spiral structure, the role of resonances.
8. The Milky Way Galaxy, the Local Group. Disk, bar, bulge and halo of the Milky Way

Pre-requisites

This course is suitable for applied mathematicians and astrophysicists. Although the course is self-contained, familiarity with Lagrangian & Hamiltonian mechanics and mathematical methods would be useful.

Preliminary Reading

1. Harwit M., 1982 *Cosmic Discovery: The Search, Scope and Heritage of Astronomy*, Basic Books
2. Elmegreen D.M., 1997 *Galaxies and Galactic Structure*, Prentice Hall
3. Sparke L., Gallagher J., 2007 *Galaxies in the Universe*, Cambridge University Press

Literature

1. Bertin G., 2000, *The Dynamics of Galaxies*, Cambridge University Press
2. Binney J., Tremaine S., 2007, *Galactic Dynamics*, Princeton University Press
3. Heggie D., Hut P. 2003, *The Million Body Problem*, Cambridge University Press
4. Murray C, Dermott S., 1999, *Solar System Dynamics*, Cambridge University Press

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Planetary System Dynamics (L24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of extrasolar planets (more than 900 are now known) and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems. Specific topics to be covered will be drawn from the following:

1. Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation
2. Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits
3. Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift
4. Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits

5. Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide
6. Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces
7. Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments
8. Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems
9. Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap

Pre-requisites

This course is self-contained.

Literature

1. Murray C. D. and Dermott S. F., *Solar System Dynamics.* Cambridge University Press, 1999.
2. Armitage P. J., *Astrophysics of Planet Formation.* Cambridge University Press, 2010.
3. de Pater I. and Lissauer J. J., *Planetary Sciences.* Cambridge University Press, 2010.
4. Valtonen M. and Karttunen H., *The Three-Body Problem.* Cambridge University Press, 2006.
5. Seager S., *Exoplanets.* University of Arizona Press, 2011.
6. Perryman M., *The Exoplanet Handbook.* Cambridge University Press, 2011.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Dynamics of Astrophysical Discs (L16)

Henrik Latter

Discs of matter in orbital motion around a massive central body occur in numerous situations in astrophysics. For example, Saturn's rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a (non-Newtonian) fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary star systems and in the centres of active galaxies; they reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of extrasolar planets.

Provisional synopsis:

- Occurrence of discs in various astronomical systems, basic physical and observational properties.
- Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.
- Viscous evolution of an accretion disc.
- Vertical disc structure, thin-disc approximations, thermal instability in cataclysmic variables.
- The shearing sheet, symmetries, shearing waves.
- Incompressible dynamics: hydrodynamic stability, vortices and dust dynamics in protoplanetary disks.
- Compressible dynamics: density waves, gravitational instability and 'gravitoturbulence' in planetary rings and protoplanetary discs.
- Satellite-disc interaction, impulse approximation, gap opening by embedded planets.
- Magnetorotational instability, 'dead zones' in protoplanetary discs.

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is helpful for the magnetorotational instability.

Literature

1. Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.
2. Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Binary Stars (L16)

Christopher Tout

A binary star is a gravitationally bound system of two component stars. Such systems are common in our Galaxy and a substantial fraction interact in ways that can significantly alter the evolution of the individual stellar components. Many of the interaction processes lend themselves to useful mathematical modelling when coupled with an understanding of the evolution of single stars.

In this course we begin by exploring the observable properties of binary stars and recall the basic dynamical properties of orbits by way of introduction. This is followed by an analysis of tides, which represent the simplest way in which the two stars can interact. From there we consider the extreme case in which tides become strong enough that mass can flow from one star to the other. We investigate the stability of such mass transfer and its effects on the orbital elements and the evolution of the individual stars. As a prototypical example we examine Algol-like systems in some detail. Mass transfer leads to the concept of stellar rejuvenation and blue stragglers. As a second example we look at the Cataclysmic Variables in which the accreting component is a white dwarf. These introduce us to novae and dwarf novae as well as a need for angular momentum loss by gravitational radiation or magnetic braking. Their formation requires an understanding of significant orbital shrinkage in what is known as common envelope evolution. Finally we apply what we have learnt to a number of exotic binary stars, such as progenitors of type Ia supernovae, X-ray binaries and millisecond pulsars.

Pre-requisites

The Michaelmas term course on Structure and Evolution of Stars is very useful but not absolutely essential. Knowledge of elementary Dynamics and Fluids will be assumed.

Literature

1. Pringle J. E. and Wade R. A., *Interacting Binary Stars.* CUP.

Reading to complement course material

1. Eggleton P. P., *Evolutionary Processes in Binary and Multiple Stars.* CUP.

Additional support

Three examples sheets will be provided and three associated two-hour examples classes will be given. There will be a two-hour revision class in the Easter Term.

Quantum Information Theory

The courses on Quantum Information Theory and Advanced Quantum Information Theory are intended to provide a comprehensive grounding in quantum information. The courses can be taken independently, but the Quantum Information Theory course provides useful background for the Advanced Quantum Information Theory course. All courses assume knowledge of basic linear algebra, probability theory, and quantum mechanics.

Desirable previous knowledge

Basic linear algebra and probability theory. Basic quantum mechanics, wave functions, amplitudes and probabilities. Dirac bra and ket formalism. Postulates of quantum mechanics, especially in the simple context of finite dimensional state spaces (state vectors, composite systems, unitary matrices, Born rule for quantum measurements).

For the Quantum Information Theory course, an additional lecture can be arranged for students who do not have the necessary background in quantum mechanics.

The following notes cover the necessary basic quantum mechanics from a quantum information theory perspective:

`http://cam.qubit.org/sites/default/files/prerequisites_13.pdf`
`http://www.qi.damtp.cam.ac.uk/sites/default/files/QCPrerequisites.pdf`

The Advanced Quantum Information Theory course assumes familiarity with basic Fourier analysis. Exposure to basic ideas of classical computer science is useful for the Advanced course, but not necessary.

The desirable previous knowledge needed to tackle the Quantum Information Theory and Advanced Quantum Information Theory courses is covered by the following Cambridge undergraduate courses. (Courses marked with an asterisk * only apply to the Quantum Information Theory course. Courses marked with a dagger † only apply to the Advanced Quantum Information Theory course.) Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year		*Courses*
First	*Essential*:	Probability, Vectors and Matrices
Second	*Essential*:	Linear Algebra, Quantum Mechanics†
	Helpful:	Quantum Mechanics*, Coding and Cryptography*, Methods† (Fourier transforms)
Third	*Very helpful*:	Linear Analysis†
	Helpful:	Principles of Quantum Mechanics

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Information Theory (M24)

William Matthews

Quantum Information Theory (QIT) is an exciting, young field which lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the mathematical theory of acquisition, storage, transmission and processing of information. QIT is the study of how these tasks can be accomplished, using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical analogues. These new features can be exploited, not only to improve the performance of certain information-processing tasks, but also to accomplish tasks which are impossible or intractable in the classical realm.

This is an introductory course on QIT, which should serve to pave the way for more advanced topics in this field. The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of QIT. Topics in this part of the course will include a brief discussion of data compression, transmission of data through noisy channels, Shannon's theorems, entropy and channel capacity.

The quantum part of the course will commence with a study of open systems and a discussion of how they necessitate a generalization of the basic postulates of quantum mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs and the Kraus Representation Theorem. Entanglement and some applications elucidating it's usefulness as a resource in QIT will be discussed. This will be followed by a study of the von Neumann entropy, its properties and its interpretation as the data compression limit of a quantum information source. Schumacher's theorem will be discussed in detail. The definition of ensemble average fidelity and entanglement fidelity will be introduced in this context. Various examples of quantum channels will be given and the different capacities of a quantum channel will be discussed. The Holevo bound on the accessible information and the Holevo-Schumacher-Westmoreland (HSW) Theorem will also be covered.

Pre-requisites

Knowledge of basic quantum mechanics will be assumed. However, an additional lecture can be arranged for students who do not have the necessary background in quantum mechanics. Elementary knowledge of Probability Theory, Vector Spaces and Linear Algebra will be useful.

Literature

I would strongly advise you to read the following notes on some fundamentals of Quantum Mechanics: `http://cam.qubit.org/sites/default/files/prerequisites_13.pdf`

The following book and lecture notes provide some interesting and relevant introductory reading material.

1. M.A.Nielsen and I.L.Chuang, Quantum Computation and Quantum Information; Cambridge University Press, 2000.
2. M.M.Wilde, From Classical to Quantum Shannon Theory, CUP; `http://arxiv.org/abs/1106.1445`.
3. J.Preskill, Chapter 5 of his lecture notes: Lecture notes on Quantum Information Theory `http://www.theory.caltech.edu/~preskill/ph229/#lecture`

Additional support

Course Instructor: Felix Leditzky

There will be four examples sheets (distributed in class) and four associated examples classes. The last examples class will be in Lent term. The course instructor will be Felix Leditzky.

Advanced Quantum Information Theory (L16)

Toby Cubitt

Quantum information theory is neither wholly physics (though it's almost entirely about quantum mechanics), nor wholly mathematics (though it's mainly concerned with proving rigorous mathematical results), nor wholly computer science (though much of it's to do with storing, processing, or transmitting information). Over the last two to three decades, it has developed into a rich mathematical theory of information in quantum mechanical systems, that draws from all three of these fields. More recently, this has been turned on its head: quantum information is beginning to be *used* to solve problems in physics, computer science, and mathematics.

The aim of this course is to select one or two advanced topics in quantum information theory, close to the cutting edge of research, and cover them in some depth and rigour.

This year, I will focus on quantum information in many-body systems. Quantum computation aims to engineer complex many-body systems to process information in ways that would not be possible classically. Many-body physics aims to understand the complex behaviour of naturally-occurring many-body systems. In a sense, they are opposite sides of the same coin. Recently, quantum information theory has been used both to prove important results in many-body physics, and to construct many-body models that exhibit very unusual physics, providing counterexamples to some of the standard intuition in condensed matter physics.

A possible outline (from which we may diverge to explore other related results) is as follows. We will begin by studying the computational complexity of quantum many-body systems, introducing the necessary complexity theoretic concepts along the way. An important milestone is Kitaev's proof of QMA-hardness of the ground state problem for local Hamiltonians. Kitaev's construction leads to systems with highly entangled ground states and polynomially-decaying spectral gap. So, in the second half of the course, we will turn to a quantum-information-inspired exploration of spectral gaps, correlations, and entanglement in many-body systems. Lieb-Robinson bounds, which limit the speed at which information can propagate in many-body systems, turn out to be a surprisingly useful tool for proving results about their static properties. We will see how they can be used to prove results relating spectral gaps, decay of correlations, and entanglement area laws (going into more or less detail, as time allows).

Pre-requisites

A solid understanding of undergraduate quantum mechanics will be assumed.

Attendance of the Michaelmas term "Quantum Information Theory" course is helpful, but not a strict requirement provided students familiarise themselves in advance with the introductory quantum information theory covered in that course. (Knowledge of Shannon theory is not necessary.)

Literature

Introductory Reading The reading material and lecture notes from the "Quantum Information Theory" course are also relevant to this course. The following cover the necessary background (and more):

1. M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press.
2. B. Schumacher and M. Westmoreland, *Quantum Processes, Systems, and Information.* Cambridge University Press.
3. John Preskill. Lecture notes on quantum information theory. `http://www.theory.caltech.edu/people/preskill/ph219/`

Reading to complement course material The first half of the course material is covered by the Kitaev book. The rest of the course material is not covered in any text book, but the following references may be helpful.

1. A. Kitaev, A. Shen and M. Vyalyi. *Classical and Quantum Computation.* American Mathematical Society
2. D. Aharonov and T. Naveh. *Quantum NP – a Survey.* `http://arxiv.org/abs/quant-ph/0210077`

3. Matt Hastings. Les Houches summer school lecture notes.
`http://arxiv.org/abs/1008.5137`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Theory and the Foundations of Physics (L8)

Non-Examinable (Graduate Level)

Adrian Kent

This non-examinable graduate course is aimed at theoretical physics Part III students, graduate students and postdocs. It may particularly interest those with interest in fundamental questions in cosmology, quantum field theory, quantum gravity or quantum information theory, as well as those interested in modern work on the foundations of quantum theory.

The exact material to be covered will depend on the audience's interests, and possibly on research developments. An indicative outline of the possible content of eight lectures is:

1. Quantum theory of closed systems. The "quantum reality problem": what exactly is the sample space on which quantum probabilities are defined? What could explain the appearance of a quasiclassical physics within a quantum universe? Bell's concept of beable. Realist versions of quantum theory.

2. Why local hidden variable theories fail: the EPR argument, Bell's theorem and generalisations.

3. Path integrals and their pitfalls. The Feynman-Hibbs argument for deriving quasiclassical physics from the path integral, and its lacunae.

4. Constraints on generalisations of quantum theory: the Pusey-Barrett-Rudolph and Colbeck-Renner theorems.

5. Could collapse be an objective physical process? Ghirardi-Rimini-Weber-Pearle models, generalisations, and experimental tests.

6. Time symmetric formulations of quantum theory.

7. A relativistically covariant solution to the quantum reality problem.

8. A survey of generalisations of quantum theory and experimental tests.

Pre-requisites

This course assumes familiarity with quantum theory. Some awareness of conceptual issues in cosmology, quantum gravity and quantum field theory would also be helpful.

Literature

The course will be self-contained. Those with time and interest might like to read John Bell's collected papers in "Speakable and Unspeakable in Quantum Mechanics" (Cambridge Univeristy Press, 2nd edition (2004)).

Additional support

The course is non-examinable; there will not be example sheets or classes.

Philosophy of Physics

The courses in Philosophy of Physics are open to all students doing Part III, but are formally listed as graduate courses. This means there is no exam at the end of May for any such course; but a Part III student can get credit for them by doing their submitted Part III essay in association with one of the courses. More generally, the Philosophy of Physics courses are intended as a refreshing and reflective companion to the other Part III courses, especially the courses in theoretical physics.

Foundations of Classical Dynamics (M8)

Non-Examinable (Graduate Level)

Nicholas J. Teh and J. Brian Pitts

Why are there two different descriptions of classical mechanics, each with its own state space geometry? How should we think about the relationship between these two descriptions? The first part of the course will survey the history and philosophy of classical mechanics, and introduce the Lagrangian and Hamiltonian descriptions before proceeding to describe the sense in which, under appropriate conditions, they describe the 'same' theory. We will also consider recent philosophical work that draws on the idea that Lagrangian and Hamiltonian descriptions are 'non-isomorphic'.

In the second part of the course we will investigate the origins of such striking claims as: (1) Hamiltonian General Relativity lacks real change, and (2) observables in Hamiltonian GR must be integrated over the universe. But how does one get from a Lagrangian to a Hamiltonian if the Lagrangian doesn't permit a Legendre transformation, as electromagnetism, Yang-Mills (weak and strong nuclear forces), and GR do not? The standard answer, Dirac-Bergmann constrained Hamiltonian dynamics, will be discussed with attention to Maxwell, a simplified form of GR discarding spatial dependence (roughly Bianchi I cosmologies), and at times Proca's massive electromagnetism (to illustrate second-class *vs.* first-class constraints). Emphasis will be placed on the equivalence of the Hamiltonian and Lagrangian formalisms, especially in relation to gauge transformations.

Pre-requisite

Familiarity with mechanics, electromagnetism, and perhaps even a bit of General Relativity might be helpful but is not required.

Literature

For the first half:

1. Eduard Zehnder, *Lectures on Dynamical Systems: Hamiltonian Vector Fields and Symplectic Capacities.*
2. Peter J. Olver, *Applications of Lie Groups to Differential Equations.*
3. Lawrence Sklar, *Philosophy and the Foundations of Dynamics.*

For the second half:

1. K. Sundermeyer, *Constrained Dynamics: With Applications to Yang–Mills Theory, General Relativity, Classical Spin, Dual String Model.*
2. J. Anderson and P. Bergmann, *Physical Review* **83** (1951), p. 1018.
3. L. Castellani, *Annals of Physics* **143** (1982), p. 357.
4. R. Wald, *General Relativity*, appendix E.
5. J. Pons and D. Salisbury, *Physical Review D* **71** (2005), 124012, gr-qc/0503013.

Additional support

An Essay will be associated with this non-examinable course.

Philosophical Aspects of Quantum Field Theory (L8)

Non-Examinable (Graduate Level)

J. Butterfield and A. Caulton

Quantum field theory has for many decades been the framework for several basic and outstandingly successful physical theories. Nowadays, it is being addressed by philosophy of physics (which has traditionally concentrated on conceptual questions raised by non-relativistic quantum mechanics and general relativity). This course will introduce this literature. More specifically, we will address the following topics: localisation and time observables in relativistic quantum theories, the renormalization group, and the algebraic approach to quantum field theory.

Pre-requisites

There are no formal prerequisites. Previous familiarity with the quantum field theory, such as provided by the Part III courses, will be helpful.

Literature

This list of introductory reading is approximately in order of increasing difficulty.

1. D. Wallace (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151** (1):33-80, 2006. Preprint available online at: http://arxiv.org/pdf/quant-ph/0112148v1
2. S. Weinberg (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: http://arxiv.org/abs/hep-th/9702027; and in Cao ed.
3. M. Fisher (1998), 'Renormalization group theory: Its basis and formulation in statistical physics', *Rev. Mod. Phys* **70**, pp 653-681.

This list of readings to complement course material is approximately in order of increasing difficulty.

1. T. Cao, (ed.) *The Conceptual Foundations of Quantum Field Theory.* Cambridge University Press, 1999.
2. L. Ruetsche, *Interpreting Quantum Mechanics.* Oxford University Press, 2011.
3. W. Greiner. *Relativistic Quantum Mechanics.* 2nd edition. Springer 1997.
4. W. Greiner and J. Reinhardt. *Field Quantization.* Springer 1996.
5. R. Haag. *Local Quantum Physics: fields, particles, algebras.* Springer 1992.
6. A. Duncan, *The Conceptual Framework of Quantum Field Theory.* Oxford University Press, 2012.

Additional support

A Part III essay will be offered in conjunction with this course.

Applied and Computational Analysis

Applied and computational analysis (ACA) is concerned with mathematical tools of broad applicability, e.g. ordinary and partial differential equations, nonlinear dynamical systems, integrable systems, numerical analysis, approximation theory, inverse problems and image analysis. While the approach is mathematical, the ultimate destination of these tools is to applications. This tension between the pure and the applied is at the core of different ACA themes.

Measure and Image (M16)

Non-Examinable (Graduate Level)

Tuomo Valkonen

Photographs and other natural images are usually not smooth maps, they contain edges (discontinuities) and other non-smooth geometric features that should be preserved by image enhancement techniques. The correct mathematical modelling of these features involves the space of functions of bounded variation and, in consequence, aspects of geometric measure theory. The aim of this course is to provide an introduction to functions of bounded variation and their applications in image processing. It will cover the following topics.

- *Motivation.* Why Sobolev spaces are not enough for image processing? Functions of bounded variation of one variable.
- *Measure.* Refresher on measure theory. Hausdorf measure and rectfiable sets.
- *Functions of bounded variation.* Weak convergence and compactness. Poincaré inequality. Co-area formula. Fine properties.
- *Total variation regularisation.* Image denoising. Basic properties of solutions.

Pre-requisites

A basic course in measure theory is strongly recommended, although we do include a quick refresher to the topic. Basic knowledge of Sobolev spaces and notions of weak convergence in function spaces are advantageous, but not necessary.

Literature

Introductory reading:

1. A. Friedman, *Foundations of Modern Analysis*, Dover, 2003.
2. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987. (Part on measure theory)
3. L. C. Evans, *Partial Differential Equations*, Americal Mathematical Society, 2010. (Part on Sobolev spaces)

Liteature to complement course material:

1. L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000.
2. G. Aubert, and P. Kornprobst, *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*, Springer, 2006.
3. L. C. Evans, and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1991.

Numerical Solution of Differential Equations (M24)

Arieh Iserles

The course will address modern algorithms for the solution of ordinary and partial differential equations, inclusive of finite difference and finite element methods, as well as broad mathematical principles underlying their analysis.

Pre-requisites

Although prior knowledge of numerical analysis and of abstract function spaces is advantageous, it will not be taken for granted. Reasonable understanding of basic concepts of analysis (complex analysis and analytic functions, basic existence and uniqueness theorems for ODEs and PDEs, elementary facts about PDEs) and of linear algebra is a prerequisite.

Literature

1. U. Ascher, *Numerical Methods for Evolutionary Differential Equations*, SIAM, 2008.
2. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations* (2nd edition), Cambridge University Press, 2006.

Additional support

An extensive printed handout, covering the entire material of the course, will be provided in the first week. There will be weekly examples' classes, starting from the third week, as well as a revision supervision in the Easter Term.

Approximation Theory (M24)

Alexei Shadrin

The course will give an overview of basic concepts of Approximation Theory, i.e., best and good approximation of a large family of functions by a smaller set (usually finitely generated, linear or nonlinear) in certain normed spaces (such as C and L_p), construction of good approximants (in various settings), finding approximation order for smooth functions.

The course consists of three parts. We start with the classic approximation by polynomials, which includes the Weierstrass theorem, positive linear operators, Chebyshev systems, and direct and inverse theorems for trigonometric approximation. We move then the emphasis to univariate splines which are piecewise polynomial functions. Here we study representation through the B-spline basis, spline interpolation theory and norm-minimization property of splines via orthogonal spline projector. Finally, we make a tour into wavelets which will cover the multiresolution analysis and Daubechies orthogonal wavelets with a compact support.

Pre-requisites

The course is mostly self-contained, but it assumes a standard mathematical analysis background (say, normed linear spaces, inner products, Fourier series) and some linear algebra. Some functional analysis tools (e.g., the Hahn-Banach theorem) appear in comments and advanced problems, so they are helpful but not required.

Preliminary Reading

To a large extent, the course follows the Lecture Notes by C. de Boor where one can find much more details on each subject. In the first half, it is also based on Cheney's book (where the most exercises are taken from).

1. C. de Boor, *Lecture Notes on Approximation Theory,* `http://www.cs.wisc.edu/~deboor`
2. E. W. Cheney, *Approximation theory,* MgGrow-Hill, New-York, 1966.

Literature

Some extracts are taken from the following books.

1. R. A. DeVore, G. G. Lorents, Constructive Approximation, Springer-Verlag, Berlin, 1993.
2. M. J. D. Powell, Approximation theory and methods, Cambridge Univeristy Press, 1981.
3. I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

Additional support

Printed handouts will be provided every lecture. Example classes will be given *each* week, i.e., after each 3 lectures, with a revision supervision in the Easter Term. There will be 2-5 exercises enclosed to each handout.

Boundary value problems for linear PDEs (L16)

Athanassios S. Fokas

Recent developments in the area of the so-called *integrable nonlinear* Partial Differential Equations (PDEs) have led to the emergence of a new method for solving boundary value problems, which is usually referred to as the *Unified Transform* (UT).

The UT will be implemented to:

(a) Linear evolution PDEs in one spatial variable formulated either on the half-line or on a finite interval. Examples include the heat equation and the Stokes equation (linearised version of the KdV).

(b) Linear elliptic PDEs in two spatial variables formulated in the interior of a convex polygon. Examples include the Laplace, the modified Helmholtz, and the Helmholtz equations.

For the above problems, in addition to presenting integral representations of the solution, simple numerical techniques for the effective computation of the solution will also be introduced.

Pre-requisites

The course only requires some elementary knowledge of complex analysis.

Literature

1. A.S. Fokas, *A unified method for boundary value problems.* 1st edition. SIAM, 2008.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Distribution Theory and Applications (L16)

A.C.L. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the use of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will look at the Sobolev spaces $H^s(\mathbf{R}^n)$ and $H^s_{\text{loc}}(X)$ and their description in terms of the Fourier transform of tempered distributions. Time permitting, the material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. In the final part of the course we will study Hörmander's oscillatory integrals.

Pre-requisites

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Analysis/Methods).

Preliminary Reading

1. Friedlander & Joshi, *Introduction to the Theory of Distributions.* Cambridge University Press, 1998.
2. Lighthill, *Introduction to Fourier Analysis and Generalised Functions.* Cambridge University Press, 1958.
3. Folland, *Introduction to Partial Differential Equations.* Princeton University Press, 1995.

Literature

1. Hörmander, *The Analysis of Partial Differential Operators: Vol I.* Springer Verlag, 1985.
2. Reed & Simon, *Methods of Modern Mathematical Physics: Vol I-II.* Academic Press, 1979.
3. Tréves, *Linear Partial Differential Equations with Constant Coefficients.* Routledge, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be available for the example sheets.

Contemporary sampling techniques and compressed sensing (L16)

Non-Examinable (Graduate Level)

Anders Hansen

This is a graduate course on sampling theory and compressed sensing for use in signal processing and medical imaging. Compressed sensing is a theory of randomisation, sparsity and non-linear optimisation techniques that breaks traditional barriers in sampling theory. Since its introduction in 2004 the field has exploded and is rapidly growing and changing. Thus, we will take the word contemporary quite literally and emphasise the latest developments, however, no previous knowledge of the field is assumed. Although the main focus will be on compressed sensing, it will be presented in the general framework of sampling theory. The course will also present related areas of sampling theory such as generalised sampling.

The course will be fairly self contained, and applications will be emphasised (in particular, signal processing, Magnetic Resonance Imaging (MRI) and X-ray Tomography). The lectures will cover the most up to date research, and although this is a Part III course, it is also aimed at Phd students and post docs who are interested in using compressed sensing and generalised sampling in their research. Students from other disciplines than mathematics are encouraged to participate.

Pre-requisites

Sampling theory and compressed sensing require a mix of mathematical tools from approximation theory, harmonic analysis, linear algebra, functional analysis, optimisation and probability theory. The course will contain discussions of both finite-dimensional and infinite-dimensional/analog signal models and thus linear algebra, Fourier analysis and functional analysis (at least basic Hilbert space theory) are important. The course will be self-contained, but students are encouraged to refresh their memories on properties of the Fourier transform as well as basic Hilbert space theory. Some basic knowledge of wavelets is useful as well as basic probability.

Preliminary Reading

For a quick and dense review of basic Fourier analysis and functional analysis chapters 5 and 8 of "Real Analysis" (Folland) are good choices. For an introductory exposition to Hilbert space theory one may use "An Introduction to Hilbert Space" (Young). And for a review of wavelets see chapters 1 and 2 of "A First Course on Wavelets" (Hernandez, Weiss). The course will cover some of the chapters of "Compressed Sensing" (Eldar, Kutyniok), so to get a feeling about the topic one may consult chapter 1 as a start.

1. Eldar, Y and Kutyniok, G., Compressed Sensing, CUP
2. Folland, G. B., Real Analysis, Wiley.
3. Hernandez, E. and Weiss, G., A First Course on Wavelets, CRC
4. Young, N., An Introduction to Hilbert Space, CUP

Literature

The following reading list complements the course material.

1. Adcock, B and Hansen, A., Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon, Appl. Comp. Harm. Anal., 32 (2012)
2. Candès, E. and Romberg, J. and Tao, T., Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory 52 (2006)
3. Donoho, D., Compressed sensing, IEEE Trans. Inform. Theory 52 (2006)

4. Körner, T. W., Fourier Analysis, CUP

5. Reed, M. and Simon, B., Functional Analysis, Elsevier

Additional support

As this is a non-examinable course there will be no examples classes, however, there will be several computer tutorials where practical implementations and real world examples will be discussed. There will also occasionally be lectures given by people from other groups outside of mathematics using compressed sensing in practice.

Homogenization of PDEs (E16)

Non-Examinable (Graduate Level)

Harsha Hutridurga

This course aims to introduce the theory of Homogenization. Partial Differential Equations with highly oscillating coefficients arise in the study of many physical phenomena (composite materials, porous media flows, rarefied gas dynamics, turbulence etc.). Homogenization, loosely speaking, replaces the PDE with highly oscillating coefficients by an equivalent PDE which "on average" behaves like the original heterogeneous PDE. All along this course, emphasis shall be given on the study of PDEs with periodically oscillating coefficients. We shall be studying the homogenization of the following PDEs:

1. Diffusion Equation: to study the conductivity of mixtures.
2. Stokes' Equation: to derive the celebrated 'Darcy's Law' in porous media.
3. Convection-Diffusion Equation: to derive the expression for 'Taylor Dispersion'.
4. Linear Boltzmann Equation: to study the interaction of the monokinetic particles with the background medium.
5. Euler Equations (incompressible): to derive the k-ε model for turbulence.

A formal method of 'Asymptotic Expansions' will be introduced in the beginning of the course followed by the more rigorous methods like the 'Energy Method' and the notion of 'Two scale Convergence'. This course shall also address the handicap of the standard Finite Element Method (FEM) to arrive at a numerical solution to PDEs with highly oscillating coefficients. As an application of the theory of Homogenization, this handicap of FEM shall be overcome by the introduction of Multiscale Finite Element Method (MFEM).

Pre-requisites

1. Some basic notions of PDEs (C. Mouhot's 'Analysis of PDE' might be useful).
2. Some compactness results from Functional Analysis (shall be recalled during the course).
3. Some basic notions of FEM (A. Iserles's 'Numerical Solution of DEs' might be useful).

Literature

1. A. Bensoussan, J.L. Lions, G.C. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.
2. D. Cioranescu, P. Donato, *An introduction to homogenization*, Oxford lecture series in mathematics and its applications 17, Oxford University Press, New York, 1999.

3. G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., Vol 23, No.6, pp.1482-1518, (1992).

4. T.Y. Hou, X-H. Wu, Z. Cai, *Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients*, Math. Comp., Vol 68, No.227, pp.913-943, (1999).

Continuum Mechanics

The four courses in the Michaelmas Term are intended to provide a broad educational background for any student preparing to start a PhD in fluid dynamics. The courses in the Lent Term are more specialized and in some cases (see the course descriptions) build on the Michaelmas Term material.

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice, familiarity with the continuum assumption, the material derivative, the stress tensor and the Navier-Stokes equation will be assumed, as will basic ideas concerning incompressible, inviscid fluid mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable. Previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is desirable for some courses. No previous knowledge of solid mechanics, Earth Sciences, or biology is required.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses

Year	*Courses*
First	Differential Equations, Dynamics and Relativity, Vector Calculus, Vectors & Matrices.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves, Asymptotic Methods.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Perturbation and Stability Methods (M24)

N.P. Peake & S.J. Cowley

The first part of this course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of the most useful mathematical tools for research will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

The second part of the course covers applications of perturbation methods to the study of fluid flows. So-called 'hydrodynamic stability' is a very broad discipline, and in this course we will concentrate on the stability of nearly parallel-flows (as for example arise in boundary-layer flows).

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. [6]
- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKBJLG' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium, including investigation of the rescaling required near 'hot spots', or'caustics'). [5]
- *The Summation of Series.* Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, Domb-Sykes plots. [1]
- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. At the end of this section further examples will be given of asymptotics beyond all orders. [6]
- *Stability Theory.* This section will review both eigenvalue and 'non-eigenvalue' aspects of stability theory as applied to fluid flows, concentrating on nearly-parallel flows. Aspects that will be covered include the concepts of 'causality' and the Briggs-Bers technique, the continuous spectrum, and the transitory algebraic growth that can follow from the fact that the operators in hydrodynamic stability theory are often not self-adjoint. [6]

Pre-requisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward differential equations and partial differential equations and evaluate simple integrals.

Literature

Introductory Reading

1. Hinch, E.J., *Perturbation Methods*, Cambridge University Press (1991).
2. Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975).

Reading to Complement Course Material

1. Bender, C.M. & Orszag, S., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). [*Beware: Bender & Orszag refer to Stokes lines as anti-Stokes lines, and vice versa.*]
2. Boyd, J.P., *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, **56**, 1-98 (1999). Also available at

`http://hdl.handle.net/2027.42/41670` and
`http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf`

3. Berry, M.V., *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).

4. Drazin, P.G. & Reid, W.H., *Hydrodynamic Stability*, Cambridge University Press (1981 & 2004).

5. Kevorkian, J. & Cole, J.D., *Perturbation Methods in Applied Mathematics*, Springer (1981).

6. Schmid, P. & Henningson, D.S., *Stability and Transition in Shear Flows*, Springer-Verlag (2001).

Additional Support

In addition to the lectures, four examples sheets will be provided and four associated examples classes will run in parallel to the course. There will be a revision class in the Easter Term.

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Desirable Previous Knowledge

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Introductory Reading

1. D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7

2. G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.

3. L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 & 5.

Reading to complement course material

1. J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).
2. S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)
3. C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).
4. O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Fluid Dynamics of the Environment (M24)

S.B. Dalziel and N.M. Vriend

Understanding and predicting the impact of human activity on the environment is a critical challenge in our time. The fluid dynamics of oceans and atmospheres plays a vital role in regulating many aspects of our Earth and our direct environment. This course introduces the basic fluid dynamics necessary to build mathematical models of the environment in which we live, and focuses on problems which occur over sufficiently small time and length scales to be largely unaffected by the earth's rotation.

The course begins by considering the governing equations of fluid flow in the presence of (typically small) density variations. "Internal gravity waves" can occur in the case of density variations in a fluid, since these variations provide a restoring force. The course highlights some of the rich and surprising dynamics of these waves. In particular, internal gravity waves radiate energy vertically as well as horizontally, and their interaction with boundaries can focus this energy and cause mixing far from where the energy was input.

Density variations within fluids can also drive the flow and the course will consider two important and related classes where the flow is either tall and thin or long and shallow. Both classes allow substantial simplification of the governing equations by integrating them over the smaller dimension. First, a relatively localised source can drive the rise of a turbulent "plume" of buoyant fluid. Volcanic eruption clouds and accidental releases of pollution are just two examples of such plumes. Second, when there are lateral gradients in fluid density interacting with horizontal or sloping boundaries, turbulent "density" or "gravity" currents can develop.

The buoyancy driving these flows may be due to differences in temperature or composition (e.g. salt or water vapour concentration), or due to the presence of a second phase such as particles or bubbles. Examples of particle-laden flows include snow avalanches, turbidity currents and pyroclastic flows. Particle-fluid and particle-particle interactions introduce a new range of interesting features. Particle suspension and deposition are important in a broad range of phenomena such as dune building and sand transport.

Pre-requisites

Desirable Previous Knowledge: Undergraduate fluid dynamics.

Literature

1. B. R. Sutherland *Internal gravity waves.* Cambridge University Press, 2010.
2. J. S. Turner, *Buoyancy Effects in Fluids.* Cambridge University Press, 1979
3. J. Pedlosky, *Geophysical Fluid Dynamics.* Springer, 1987

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Biological Physics (M24)

R.E. Goldstein & U. Keyser

This course will provide an overview of the physics and mathematical description of living systems. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology. The topics to be covered will span the range of length scales from molecular to ecological, with emphasis on key paradigms. Introductory material on statistical mechanics will provide background for much of the course. The subsequent topics will include *Microscopic Physics* – van der Waals forces, screened electrostatics, Borwnian motion, fluctuation-dissipation theorem; *Fluctuation-Induced Forces* – polymer physics, random walks, entropic forces, stiff chains, self-avoidance, dynamics, protein folding; *Elasticity* – differential geometry of curves and suraces, linear elasticity theory, thin plates and rods, Helfrich model for membranes, elastohydrodynamics; *Chemical Kinetics and Pattern Formation*– Michaelis-Menten kinetics, oscillations, excitable media, ion channels, action potentials, reaction-diffusion dynamics, Fitzhugh-Nagumo model, spiral waves; *Dynamics*– life at low Reynolds numbers, chemoreception, advection-diffusion problems.

Pre-requisites

Some familiarity with statistical physics will be helpful.

Preliminary Reading

1. P. Nelson. *Biological Physics.* W.H. Freeman (2007).
2. J.D. Murray. *Mathematical Biology I. & II.* Springer (2007, 2008).
3. K. Dill & S. Bromberg. *Molecular Driving Forces.* Garland (2009).

Literature

1. B. Alberts, A. Johnson, J. Lewis, M. Raff, K. Roberts and P. Walter. *Molecular Biology of the Cell.* 5th edition. Garland Science (2007).
2. J.N. Israelachvili. *Intermolecular and Surface Forces.* 2nd edition. Academic Press (1992).
3. E.J.W. Verwey and J.Th.G. Overbeek. *Theory of the Stability of Lyophobic Colloids.* Elsevier (1948).
4. M. Doi and S.F. Edwards. *The Theory of Polymer Dynamics.* OUP (1986).
5. A. Parsegian. *Van der Waals Forces.* CUP (2005).
6. D. Andelman & W. Poon. *Condensed Matter Physics in Molecular and Cell Biology.* Taylor & Francis (2006).

Additional support

4 examples sheets and 4 examples classes will be scheduled, along with a revision class in Easter term.

Direct and Inverse Scattering of Waves (L16)

Orsola Rath Spivack

The study of wave scattering is concerned with how the propagation of waves is affected by objects, and has a variety of applications in many fields, from environmental science to seismology, medicine, telecommunications, materials science, military applications, and many others. If we know the nature of the objects and we want to find how an incident wave is scattered, we call this a 'direct scattering problem' and practical applications will include for example underwater sound propagation, light transmission through the atmosphere, or the effect of noise in built-up areas. If we measure and know the scattered field produced by an incident wave, but we do not know the nature of the objects that have scattered it, we call this an 'inverse scattering problem' and applications will include for example non-destructive testing of materials, remote sensing with radar or lidar, or medical imaging.

This course will provide the basic theory of wave propagation and scattering and an overview of the main mathematical methods and approximations, with particular emphasis on inhomogeneous and random media, and on the regularisation of inverse scattering problems. Only time-harmonic waves will be normally considered.

Topics covered will include:

1. Boundary value problems and the integral form of the wave equation.
2. The parabolic equation and Born and Rytov approximations for the scattering problem.
3. Scattering by randomly rough surfaces and propagation in inhomogeneous media.
4. Ill-posedness of the inverse scattering problem, and the Moore-Penrose generalised inverse.
5. Regularisation methods and methods for solving some inverse scattering problems.
6. Time reversal and focusing in inhomogeneous media.

Students considering this course might also like to consider the complementary course "Sound Generation and Propagation", and courses on Imaging.

Pre-requisites

This course assumes basic knowledge of ODEs and PDEs, and of Fourier transforms. Some familiarity with linear algebra and with basic concepts in functional analysis is helpful, though by no means necessary.

Literature

Introductory Reading

1. C.W. Groetsch, *Inverse Problems in the Mathematical Sciences.* Braunschweig 1993
2. L.D. Landau and E.M. Lifschitz, *Fluid Dynamics.* Pergamon [Chapter 8]

Reading to complement course material

1. D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, 1992.
2. D.G. Crighton et al, *Modern Methods in Analytical Acoustics.* Springer, 1992.
3. H.W. Engl, M. Hanke and A. Neubauer, *Regularization of inverse problems.* Kluwer, 2000.
4. A. Ishimaru, *Wave Propagation and Scattering in Random Media.* Academic Press, 1978.
5. A. Kirsch, *An introduction to the mathematical theory of inverse problems.* Springer, 1996.
6. B. Uscinski, *The elements of wave propagation in random media.* McGraw-Hill, 1977.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Sound Generation and Propagation (L16)

Ed Brambley

The application of wave theory to problems involving the generation, propagation and scattering of acoustic and other waves is of considerable relevance in many practical situations. These include, for example, underwater sound propagation, aircraft noise, remote sensing, the effect of noise in built-up areas, and a variety of medical diagnostic applications. This course aims to provide the basic theory of wave generation, propagation and scattering, and an overview of the mathematical methods and approximations used to tackle these problems, with emphasis on applications to aeroacoustics. The course will cover some general aeroacoustic theory [3], sound generation by turbulence and moving bodies (including the Lighthill and Ffowcs Williams–Hawkings acoustic analogies) [3], scattering (including the scalar Wiener-Hopf technique applied to the Sommerfeld problem of scattering by a sharp edge) [4], long-distance sound propagation including nonlinear and viscous effects [3], and wave-guides [3].

Pre-requisites

This course assumes that students have attended some introductory courses in continuum mechanics and complex variable theory (especially Fourier transforms and their inversion using complex residues). Attendance at the Part III course Perturbation and Stability Methods would also be helpful, but is by no means essential.

Literature

1. Landau, LD and Lifschitz, EM. *Fluid Mechanics*, Butterworth-Heinemann. [Chapters 1 & 8]
2. Hinch, EJ. *Perturbation Methods*, CUP. [Chapters 3 & 7]

Additional support

Three examples sheets will be provided and three associated two-hour examples classes will be given. There will be one two-hour revision class in the Easter Term.

Fluid dynamics of the solid Earth (L16)

Jerome A. Neufeld, M. Grae Worster

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth's core is formed by the solidification of a mixture of molten iron and various lighter elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth's magnetic field. At very much longer time scales, radiogenic heating of the solid mantle drives solid-state convection resulting in plume-like features possibly responsible for features such as the Hawaiian sea mounts. Nearer the surface, convection drives the motion of brittle plates which are responsible for the Earth's topography as can be felt and imaged through the seismic record. Upwelling mantle material also drives partial melting of mantle rocks resulting in compaction, and ultimately in the propagation of viscous melt through the elastic crust. On the Earth's surface, and at very much faster rates, the same physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of sea ice to the flow of glacial ice.

This course will use the wealth of observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials.

Desirable Previous knowledge

A basic understanding of viscous fluid dynamics. Mathematical methods, particularly the solution of ordinary and partial differential equations.

Literature

1. M.G. Worster. *Solidification of Fluids.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
2. H.E. Huppert. *Geological fluid mechanics.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
3. D.L. Turcotte, G. Schubert. Geodynamics, second edition. CUP (2002)

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Fluid Dynamics of Climate (L24)

P.F. Linden & J.R. Taylor

Understanding and predicting the Earth's climate is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the Earth's climate, helping to make the planet hospitable for life. However, the dynamical complexity of this motion and the wide range of space and time scales involved, makes predicting the climate system a very difficult endeavour.

This course provides an introduction to the basic fluid dynamics necessary to build mathematical models of the environment in which we live, focusing on the large-scale behaviour of stratified and rotating flows. The course begins by considering flows where the timescale for the motion is long compared with a day and the Earth's rotation plays an important role. The additional timescale introduced by the Earth's rotation modifies the dynamics in a profound way for both homogeneous and density stratified flows. The Coriolis force (a fictitious force arising from our use of a frame of reference rotating with the planet) causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics. We will then apply the theory for rotating, stratified fluids to describe the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the dynamics that give rise to eddies and storms in the ocean and atmosphere, ocean gyres and boundary currents like the Gulf Stream, the meridional (north/south) circulation in the ocean and atmosphere, and the transport of heat and other tracers across the globe.

Desirable Previous Knowledge

Undergraduate fluid dynamics

Reading to complement course material

1. Gill, A.E., Atmosphere-Ocean Dynamics. Academic Press (1982).
2. Vallis, G.K. Atmospheric and Oceanic Fluid Dynamics. Cambridge University Press. (2006).
3. Pedlosky, J. Geophysical Fluid Dynamics. Springer. (1987).
4. Marshall, J. and R.A. Plumb. Atmosphere, Ocean, and Climate Dynamics. Academic Press. (2008).
5. J.S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Complex and Biological Fluids (L24)

Dr Eric Lauga

Fluid mechanics plays a crucial role in a number of biological processes, from the largest of animals to the smallest of cells. In this course, we will give an overview of the hydrodynamic phenomena associated with biological life at the cellular scale, from the fluid mechanics of individual microorganisms and their appendages to the modelling of collective, complex, cell dynamics. We will combine physical description, scaling analysis, and detailed calculations in order to present a wide overview of the subject, and appeal to students in applied mathematics, physics, and quantitative biology.

In the first part of the course, we will introduce the fluid dynamics and soft matter mechanics relevant to the locomotion of individual cells. Drawing examples from a variety of organisms, we will aim at providing a precise mathematical description of how cells actuate and exploit surrounding fluids in order to self-propel, and how they interact with their chemical and mechanical environment. The second part of the course will review classical models for the flow of complex fluids and will then build on them in order to derive the framework describing the collective dynamics of populations of interacting cells.

At the end of the course, students will be equipped to carry out independent research in biological physics and fluid dynamics relevant to the cellular world.

Pre-requisites

Undergraduate fluid dynamics, vector calculus and mathematical methods. Attendance to Part III "Slow Viscous Flows" is required and Part III "Biological Physics" is recommended.

Literature

1. National Committee for Fluid Mechanics Films on "Rheological Behavior of Fluids" and "Low Reynolds Number Flow" at: `http://web.mit.edu/hml/ncfmf.html`
2. Lighthill (1975) Mathematical Biofluiddynamics, SIAM.
3. Purcell (1977) Life at low Reynolds number. *American Journal of Physics*, **45**, 3-11.
4. Childress (1981) Mechanics of Flying and Swimming, CUP.
5. Berg (2000) Motile Behavior of Bacteria. *Physics Today*, **53**, 24.
6. Bray (2000) Cell Movements, Garland.
7. Morrison (2001) Understanding Rheology, OUP.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Demonstrations in Fluid Mechanics (L8)

Non-Examinable (Graduate Level)

Dr. S.B. Dalziel, Dr. J. Neufeld

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- porous media and carbon sequestration;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

Desirable Previous Knowledge

Undergraduate Fluid Dynamics.

Reading to complement course material

1. M. Van Dyke. An Album of Fluid Motion. Parabolic Press.
2. G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.
3. M. Samimy, K. Breuer, P. Steen, & L. G. Leal. A Gallery of Fluid Motion. CUP.