UNIVERSITY OF CAMBRIDGE
Faculty of Mathematics

Mathematical Tripos
Part III Guide to Courses 2013-2014

The Faulkes Institute of Geometry, completed in January 2002

Mathematical Tripos
Part III Lecture Courses in 2013-2014

Department of Pure Mathematics & Mathematical Statistics

Department of Applied Mathematics & Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*, and are indicated as such after the title. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2013-14. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures and associated course materials as offered in this academic year define the syllabus. Each course lecturer has discretion to vary the material covered.
- This document was last updated in September 2013. Further changes to the list of courses will be avoided if at all possible, but may be essential, and will appear in the online version at available at http://www.maths.cam.ac.uk/postgrad/mathiii/
- Some graduate courses have no writeup available. Hopefully, the title of the course is sufficiently explanatory.

Table of Contents

Algebra

Commutative Algebra (M24)

C.J.B. Brookes

The aim of the course is to give an introduction to the theory of commutative Noetherian rings and modules, a theory that is an essential ingredient in algebraic geometry, algebraic number theory and representation theory.

Topics I hope to fit in will be the theory of ideals for Noetherian and Artinian rings; localisations and completions; integral closure, valuation rings and Dedekind rings; dimension theory; projective and injective modules, resolutions, Koszul complex, (co)homology, derivations and Kaehler differentials.

There will be four example sheets.

Desirable Previous Knowledge

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields will be helpful but not necessary.

Books

1. M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra, Addison-Wesley, 1969.
2. N. Bourbaki, Commutative algebra, Elements of Mathematics, Springer, 1989.
3. H. Matsumura, Commutative ring theory, Cambridge Studies 8, Cambridge University Press, 1989.
4. M.Reid, Undergraduate Commutative Algebra, LMS student texts 29, Cambridge University Press, 1995.
5. R.Y. Sharp, Steps in commutative algebra, LMS Student Texts 19, Cambridge University Press, 1990.

The basic text is Atiyah and Macdonald but it doesn't go into much detail and many results are left to the exercises. Sharp fills in some of the detail but neither book goes far enough. Matsumura covers the additional homological material but is a bit tough as an introduction. Reid's book is a companion to one on algebraic geometry and that influences his choice of topics and examples. Bourbaki is encyclopaedic.

Computational Group Theory (M24)

Non-Examinable (Graduate Level)

Richard Parker

Computational Group Theory is the study of ways of using a computer to solve problems in group theory, a thriving and active area at present, frequently posing questions that lead, rather than follow, theoretical work.

This course needs little preparation above an undergraduate level of algebra and some knowledge of computing.

The course will first cover much of the "classical" computational group theory, including the various algorithms for working with permutations, rewriting (which will also include Groebner Bases) and working with matrices over finite fields. This part will encompass about the first twelve lectures.

The second half of the course will cover the more recent work, including working with matrices of integers (to represent the group, or otherwise), the (very) current Matrix recognition project (what group do these matrices generate?) and Small Cancellation methods.

The nature of modern computers (multi-core, memory bound) make further demands on these algorithms and ideas will be discussed as we go along as to how these algorithms need to be updated to allow for these demands.

Literature

No preliminary reading is required, but the following two books present (between them) a considerable amount of the course.

1. Computation with Finitely Presented Groups Charlie C Sims C.U.P. 1994

2. Handbook of Computational Group Theory Derek F Holt, Bettina Eick and Eamonn OBrien Chapman+Hall/CRC 2005

Lie algebras and their representations (M24)

D.I. Stewart

Lie algebras were introduced by Sophus Lie as a way to study what we now call Lie Groups. The latter can be thought of as smooth groups. Then Lie algebras arise by looking at infinitesimal transformations, specifically, of the tangent space at the identity. We'll go through these concepts in some detail, but actually the definition of a Lie algebra (which will be given in approximately three lines) is simply a vector space with a certain anticommutative multiplication which satisfies some version of associativity. So for the most part, all the geometry of the Lie group can be exorcised and we can get down to the algebraic arguments which will give us a complete picture of the finite-dimensional complex representations of finite-dimensional semisimple Lie algebras. But we'll do more than that, giving a classification of the complex simple Lie algebras by root data, covering all the structure theory necessary to get us there.

Lie theory comes in many flavours and is important in finite group theory (with 26 exceptions all nonabelian finite simple groups come from Lie theoretic objects), number theory (notably the Langlands programme), physics (e.g. quantum), differential equations, integrable systems ... Underpinning all Lie theoretical objects are root systems. In some way this course can be seen as an introduction to those most fundamental of mathematical objects, as motivated by Lie algebras.

Desirable Previous Knowledge

You need to be happy with the notion of a vector space but that's more-or-less it. I'm planning to illustrate many of the theorems by showing how they go wrong over fields of positive characteristic, so a basic familiarity with the existence of such fields would be good. Having taken some course on representation theory in the past would be a plus, only so that terms like 'completely reducible' are familiar.

Reading to complement course material

1. Representation theory, Fulton and Harris. Springer. This is a beautiful book written in a fun, chatty style with plenty of examples, motivation, and pictures. It tells a good story. It is the main source of the lecture notes and would be a great complement to the course. It also has stuff on representations of the symmetric groups. If you are thinking of staying on in algebra, it would be a great purchase.

2. Introduction to Lie algebras and representation theory. Humphreys. Springer. A good book, taking a more algebraic approach.

3. Introduction to Lie algebras. Erdmann and Wilson. Springer. Again more algebraic while a little more accessible than the Humphreys.

Representations and quivers (L24)

Stuart Martin

Quivers are very simple mathematical objects: finite directed graphs. A representation of a quiver assigns a vector space to each vertex, and a linear map to each arrow. Quiver representations were originally introduced to treat problems of linear algebra, for example, the classification of tuples of subspaces of a prescribed vector space. But it soon turned out that quivers and their representations play an important role in representation theory of fnite-dimensional algebras; they also occur in less expected domains of mathematics including Kac-Moody Lie algebras, quantum groups, Coxeter groups, and geometric invariant theory. This course presents some fundamental results and examples of quiver representations, in their algebraic and geometric aspects. Our main goal is to give an account of a theorem of Gabriel characterizing quivers of finite orbit type, that is, having only finitely many isomorphism classes of representations in any prescribed dimension: such quivers are exactly the disjoint unions of Dynkin diagrams of types A_n, D_n, E_6, E_7, E_8, equipped with arbitrary orientations. Moreover, the isomorphism classes of indecomposable representations correspond bijectively to the positive roots of the associated root system. This beautiful result has many applications to problems of linear algebra. For example, when applied to an appropriate quiver of type D_4, it yields a classification of triples of subspaces of a prescribed vector space, by finitely many combinatorial invariants. The corresponding classification for quadruples of subspaces involves one-parameter families (the "tame" case); for r-tuples with $r \geq 5$ one obtains families depending on an arbitrary number of parameters (the "wild" case).

Gabriel's theorem holds over an arbitrary field; in the course, we only consider algebraically closed fields, in order to keep the prerequisites at a minimum.

- Quivers, representations, path algebras; examples
- Module and cohomology theory
- Review/introduction to algebraic groups;
- The representation variety
- Introduction to Euclidean and Dynkin diagrams
- Representation type, Gabriel's Theorem.
- Representations of finitely-generated algebras
- Research topic (if time permits): maybe Donkin's work from [4] below.

Two sheets of examples will be provided backed up by two classes.

Desirable Previous Knowledge

Prerequisites are fairly modest: basic notions about rings and modules; a little homological algebra (up to and including Ext^1 and the long exact sequence); some algebraic geometry (Zariski topology on affine space, dimension, morphisms, Zariski tangent spaces, differentials, varieties, affine schemes from [5], [6]), basic category theory.

Introductory Reading

1. I. Assem, A. Skowronski and D. Simson, Elements of the Representation Theory of Associative Algebras I: Techniques of Representation Theory, London Math. Soc. Student Texts 65 (CUP, 2006).
2. M. Auslander, I. Reiten and S. O. Smalo, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Math. 36 (CUP, 1995).

3. D. J. Benson, Representations and cohomology I: Basic representation theory of finite groups and associative algebras, Cambridge Studies in Advanced Math. 30, (CUP, 1991).
4. S. Donkin, Polynomial invariants of representations of quivers, Comment. Math. Helv. 69 (1994), 137–141
5. D. Eisenbud, Commutative algebra with a view towards algebraic geometry, Graduate Texts in Math. 150, Springer-Verlag, New York (1995).
6. D. Eisenbud, J. Harris, The geometry of schemes, Graduate Texts in Math. 197, Springer-Verlag, New York (2000).

Reading to complement course material

1. H. Derksen and J. Weyman, Quiver representations, Notices Amer. Math. Soc. 52 (2005), 200–206.
2. M. Geck, An introduction to algebraic geometry and algebraic groups, Oxford Graduate Texts in Mathematics (OUP, 2003).
3. C. Ringel, Four papers on problems in linear algebra, in: Representation theory, 141–156, London Math. Soc. Lecture Note Ser. 69, (CUP, 1982).
4. T. A. Springer, Linear Algebraic Groups, Second edition, Progress in Math. 9, (Birkhauser, Basel, 1998).

Topics in Infinite Groups (L16)

Jack Button

This course is a general introduction to infinite groups, with particular emphasis on finitely generated and finitely presented groups.

The following is a summary of the lectures:

Review of basic definitions and results; Brief mention of Abelian groups.

Free (non-abelian) groups and free products, Nielsen-Schreier theorem and index formula; Presentations of groups, Free products with amalgamation and HNN extensions; nilpotent, polycyclic and soluble groups.

Subgroups of finite index and virtual properties; maximal and maximal normal subgroups; infinite simple groups; residual finiteness and Hopficity; Baumslag-Solitar groups.

The generalised Burnside Problem.

Desirable Previous Knowledge

Any introductory undergraduate group theory course as well as some basic algebraic topology, up to covering spaces and the fundamental group.

Introductory Reading

Any introductory text in group theory, of which there are plenty. To list but two:

1. J. F. Humphreys, *A course in group theory*, Oxford Science Publications
2. W. Ledermann, *Introduction to group theory*, Longman

The necessary algebraic topology can certainly be found in either of:

1. J. M. Lee, *Introduction to topological manifolds*, (GTM 202), Chapters 7, 10, 11, 12
2. A. Hatcher, *Algebraic topology*, CUP, Chapter 0 and Sections 1.1, 1.2, 1.3, 1.A

Reading to complement course material

Some of the results from the course can be found in:

1. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer, Sections I.1, II.1, II.2, IV.1, IV.2, IV.4

Analysis

Analysis of Partial Differential Equations (M24)

Clément Mouhot

The purpose of this course is to introduce some techniques and methodologies in the mathematical treatment of Partial Differential Equations (PDE). The theory of PDE is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th century. It is at a crossroad with physics and many areas of pure and applied mathematics.

The course begins with an introduction to four prototype linear equations: Laplace's equation, the heat equation, the wave equation and Schrödinger's equation. Emphasis will be given to the modern functional analytic techniques relying on the notion of Cauchy problem and estimates rather than explicit solutions, although the interaction with classical methods (e.g. the fundamental solution, Fourier representations) will be discussed. The following basic unifying concepts will be studied: well-posedness, energy estimates, elliptic regularity, characteristics, propagation of singularities, group velocity, and the maximum principle. The course will end with a discussion of some of the open problems in PDE.

Pre-requisite Mathematics

There are no specific pre-requisites beyond a standard undergraduate analysis background; in particular a familiarity with measure and integration theory is useful. The course will be mostly self-contained and can be used as a first introductory course in PDE for students wishing to continue with some specialised PDE Part III courses in lent and easter terms (elliptic PDE, kinetic PDE, PDE and image processing. . .). In particular having attended the "Partial differential equations" course in Part II is useful but is *not* a pre-requisite.

Literature

Some lecture notes are available online at `http://cmouhot.wordpress.com/teachings/`.

The following textbooks are excellent references:

Evans, L. C., *Partial Differential Equations*, Springer, 2010.

John, F., *Partial Differential Equations*, Springer, 1991.

The following review gives an overview of the field of PDE:

Klainerman, S., *Partial Differential Equations*, Princeton Companion to Mathematics (editor T. Gowers), Princeton University Press, 2008.

Additional Information

This course is also intended for doctoral students of the Centre for Analysis (CCA), who will also be involved in additional assigments, presentations and group work. Part III students do not do these, and they will be assessed in the usual way by exam at the end of the academic year.

Functional Analysis. (M24)

András Zsák

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. We begin with a review of some of the material of the Part II Linear Analysis course which will be taken for granted (see prerequisites below). We then cover the following topics:

Hahn-Banach Theorem.

Riesz Representation Theorem.

Weak and weak-* topologies. Theorems of Mazur, Goldstein, Banach-Alaoglu.

Locally convex spaces, separation of convex sets. Extreme points and the Krein-Milman theorem.

Banach algebras, spectral theory. Commutative Banach algebras and the Gelfand representation theorem. Holomorphic functional calculus.

Hilbert space operators, C^*-algebras. The Gelfand-Naimark theorem. Spectral theorem for commutative C^*-algebras. Spectral theorem and Borel functional calculus for normal operators.

Some additional topics time permitting.

There will be a number of Examples Sheets and Examples Classes during the term. For the latter, time and place will be arranged during lectures.

Desirable Previous Knowledge

Thorough grounding in basic topology and analysis.

Some knowledge of basic functional analysis. Specifically, the following results will be taken for granted (although, they will be recalled and, for some, proofs will be given): definition and examples of normed spaces and bounded linear operators; operator norm; equivalence of norms on finite-dimensional normed spaces; finite-dimensional subspace of normed space is closed; Baire Category Theorem, Open Mapping Lemma, Open Mapping Theorem, Closed Graph Theorem; Stone-Weierstrass Theorem; Urysohn's Lemma; Arzela-Ascoli theorem. Hilbert spaces; orthogonal decompositions; orthonormal bases; Riesz Representation Theorem (the one identifying the dual of Hilbert space); adjoint operators.

In the section on the Riesz Representation Theorem and in Spectral Theory, some knowledge of measure theory will be very useful but not entirely essential.

In Spectral Theory we will make use of basic complex analysis, for example, Cauchy's Integral Formula, Maximum Modulus Principle.

Introductory Reading

The first two books are excellent both for introductory reading and for the course. If you want to brush up on your measure theory, I really like the third book but, of course, there are plenty of others on the subject.

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales.* Oxford University Press, 2011. ISBN: 9780199206537, 9780199206544. Series title: Oxford graduate texts in mathematics 20. Available in Betty & Gordon Moore Library (QA322.2.A45 2011).

2. Bollobás, Béla *Linear analysis : an introductory course.* Cambridge University Press, 1990. ISBN: 0521383013, 0521387299. Available in Betty & Gordon Moore Library (QA320.B65 1990) and University Library South Front, Floor 4 (349:4.c.95.416).

3. Taylor, S. J. *Introduction to measure and integration.* Cambridge University Press 1973. ISBN: 0521098041. Available in Betty & Gordon Moore Library (QA312.T39 1973) and University Library South Wing, Floor 5 (202.c.97.337).

Reading to complement course material

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales* (see above).

2. Bollobás, Béla *Linear analysis : an introductory course* (see above).

3. Garling, D. J. H. *A journey into linear analysis.* Cambridge University Press, 2007. ISBN: 9780521876247, 9780521699730. Available in Betty & Gordon Moore Library (QA295.G375 2007).

4. Murphy, Gerard J. *C∗-algebras and operator theory.* Academic Press, 1990. ISBN: 0125113609. Available in Betty & Gordon Moore Library (QA326.M877 1990) and University Library South Front, Floor 4 (350:9.c.95.331).

5. Rudin, Walter *Real and complex analysis.* McGraw-Hill, 1987. ISBN: 0071002766, 0070542341. Available in Betty & Gordon Moore Library (QA300.R83 1987).

6. Rudin, Walter *Functional analysis.* McGraw-Hill, 1991. ISBN: 9780070542365, 0070542368. Available in Betty & Gordon Moore Library (QA320.R83 1991, QA320.R83 1991).

Topics in Kinetic Theory (L24)

Amit Einav and Chanwoo Kim

Description

Kinetic equation are a particular type of, usually non linear, Partial Differential Equations (PDEs) that arise in Statistical Physics. Their goal is to describe the time evolution of systems consisting of large amount of objects, such as Plasmas, Galaxies and Dilute Gases. This course is an introductory course to the modern analysis of kinetic equations, aiming to present some results on the fundamentally important Boltzmann equation from the subject of gas dynamics.
The course is suitable for both Pure Mathematics and Applied Mathematics students. We hope to cover the following topics:

1. Introduction:
   - Microscopic, Macroscopic and Mesoscopic Viewpoints and Kinetic Theory.
   - From ODEs to PDEs.
2. Derivation of Kinetic Equations:
   - Newtonian and Statistical Viewpoints.
   - The Characteristic Method.
   - The Many Particle Limit and Mean Field Models.
3. Linear Transport Equations:
   - Lagrangian and Eulerian Viewpoints.
   - Dispersion Estimations.
   - Averaging Lemma and Phase Mixing.
4. The Linear Boltzmann Equation:
   - A Probabilistic Interpretation.
   - The Cauchy Theory.
   - The Maximum Principle.
   - Relaxation to Equilibrium.
5. Additional Topics.

Pre-requisite Mathematics

Knowledge of basic Measure Theory, Functional Analysis and simple methods in Ordinary Differential Equations (as in the 1A course 'Differential Equations') is required. Any advanced knowledge in the above topics, as well as knowledge in PDEs, Sobolev spaces and Fourier Analysis, can benefit the student, but is not mandatory. Students are welcome to discuss any pre-requisite requirements with the Lecturers prior to the beginning of the course.

Literature

The course is mainly self contained and requires no textbook. However, there are numerous textbooks that will compliment the material of the course, or help bring the student up to pace with the pre requisites of it. Interested students are welcome to discuss this with the Lectures.

The Strong Maximum Principle for Singular Minimal Hypersurfaces and Related Topics (M24)

Non-Examinable (Graduate Level)

Brian Krummel and Neshan Wickramasekera

If two connected, smoothly embedded minimal hypersurfaces (i.e. critical points of hypersurface-area) in a Riemannian manifold have the property that near each of their common points, one hypersurface lies locally on one side of the other, then the hypersurfaces are either disjoint or they coincide. This is the strong maximum principle for smooth minimal hypersurfaces, and it is an easy consequence of the Hopf maximum principle for second order, linear elliptic PDEs.

Just as with many situations of solutions to non-linear variational problems, minimal hypersurfaces need not be smooth everywhere. A natural question is whether the above strong maximum principle extends to singular minimal hypersurfaces. While the answer in general is no, it is known to be yes under various additional hypotheses. There is an extremely rich theory surrounding this question, and the course will aim to cover as much of it as possible, hopefully ending with a very recent result giving a sharp condition under which the maximum principle holds.

Pre-requisite Mathematics

A good background in measure theory, linear elliptic PDE and differential geometry of hypersurfaces.

Literature

1. David Gilbarg and Neil S. Trudinger, "Elliptic Partial Differential Equations of Second Order." Springer-Verlag (1983).

2. Leon Simon, "Lectures on Geometric Measure Theory." Proceedings of the Centre for Mathematical Analysis, Australian National University, Vol. 3, (1983).

3. Bombieri, E., and E. Giusti. "Harnack's inequality for elliptic differential equations on minimal surfaces." Inventiones Mathematicae 15.1 (1972): 24-46.

4. Simon, Leon. "A strict maximum principle for area minimizing hypersurfaces." Journal of Differential Geometry 26.2 (1987): 327-335.

5. Solomon, Bruce, and Brian White. "A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals." Indiana University Mathematics Journal 38.3 (1989): 683-691.

6. Ilmanen, T. "A strong maximum principle for singular minimal hypersurfaces." Calculus of Variations and Partial Differential Equations 4.5 (1996): 443-467.

7. Wickramasekera, Neshan. "A general regularity theory for stable codimension 1 integral varifolds." Annals of Math. (2013), to appear. (also available at: arXiv:0911.4883).

8. Wickramasekera, Neshan. "A sharp strong maximum principle and a sharp unique continuation theorem for singular minimal hypersurfaces." preprint, (2013).

Analysis on Polish spaces (L24)

Dr D.J.H. Garling

A Polish space is a topological space homeomorphic to a complete separable metric space, and it is where most analysis takes place. Some or all of the following topics will be considered.

Isoperimetry in Euclidean space and in spheres.

Gaussian isoperimetry.

Feller semigroups: the heat, Ornstein-Uhlenbeck and Poisson semigroups.

Energy and entropy: Poincaré and logarithmic Sobolev inequalities.

SubGaussian random variables.

Convergence of measures, and optimal transportation.

Concentration of measure.

Desirable Previous Knowledge

Basic knowledge of analysis and general topology. Results from the the Part II Linear Analysis and Probability and Measure courses will be used, but detailed knowledge of their proofs will not be needed. Similarly, results from Dr Zsak's course may be used.

Introductory Reading

1. R.M.Dudley. Real analysis and probability (Chapters 1-5). CUP 2002.

Reading to complement course material

1. R.M.Dudley. Real analysis and probability (Chapter 11). CUP 2002.
2. D.J.H. Garling. Inequalities (Chapter 13). CUP 2007
3. Jerome A. Goldstein. Semigroups of linear operators and applications. OUP 1985.

Further reading

1. C.Ané et al. Sur les inégalités de Sobolev logarithmiques. SMF 2000.
2. N. Berestycki and R.Nickl. Concentration of measure. On their websites. 2009.
3. M. Ledoux. The concentration of measure phenomenon. AMS 2001.
4. C. Villani. Topics in Optimal transportation. AMS 2003.
5. C. Villani. Optimal transport old and new. Springer 2009.

Function Spaces (L24)

Non-Examinable (Graduate Level)

Sophia Demoulini

Following a short revision of weak topologies in Banach spaces, weak compactness and convergence theory, we study Sobolev spaces, Hardy spaces, BMO and VMO theory and also BV functions if time allows.

Prerequisites

Measure space and L^p space theories, basic Banach and Hilbert spaces, and duality in weak topologies.

Literature

1. Brezis, *Functional Analysis and PDE*, Springer.
2. Evans, *Measure Theory and Fine Properties of Functions*, CRC Press.
3. Stein, *Harmonic Analysis*, Chapters III and IV, Princeton University Press.

Calculus of Variations (E16)

Non-Examinable (Graduate Level)

Sophia Demoulini

The direct method in the calculus of variations, lower semi-continuity and quasiconvexity. Generalized notions of convexity. Relaxation and minimization of functionals. Useful tools such as Young measures will be introduced.

If time allows we will also look at minimization in L^1, the Dunford-Pettis theorem, weak compactness and convergence in L^1, compensated compactness and concentration.

Prerequisites

Measure theory, L^p and Sobolev spaces, including weak convergence and compactness.

Literature

1. Dacorogna, *The Direct Method in the Calculus of Variations*, Springer.
2. Ekeland and Temam, *Analyse convexe et problèmes variationnels* , Dunod.
3. Fonseca and Leoni, *Modern methods in the calculus of variations: L^p spaces*, Springer.

Combinatorics

Ramsey Theory (M16)

I. B. Leader

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. A typical example is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions.

The flavour of the course is combinatorial. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. We shall cover a number of 'classical' Ramsey theorems, such as Gallai's theorem and the Hales-Jewett theorem, as well as some more recent developments. There will also be several indications of open problems.

We hope to cover the following material.

Monochromatic Systems

Ramsey's theorem (finite and infinite). Canonical Ramsey theorems. Colourings of the natural numbers; focusing and van der Waerden's theorem. Combinatorial lines and the Hales-Jewett theorem. Applications, including Gallai's theorem.

Partition Regular Equations

Definitions and examples. The columns property; Rado's theorem. Applications. (m, p, c)-sets and Deuber's theorem. Ultrafilters; the Stone-Čech compactification. Idempotent ultrafilters and Hindman's theorem.

Infinite Ramsey Theory

Basic definitions. Not all sets are Ramsey. Open sets and the Galvin-Prikry lemma. Borel sets are Ramsey. Applications.

Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey's theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness.

Appropriate books

1. B. Bollobás, *Combinatorics*, C.U.P. 1986
2. R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, John Wiley 1990

Algebraic methods in incidence theory (L16)

Michael Bateman

The goal of this course is to study applications of the polynomial method and of Bezout's theorem.

The polynomial method is, loosely speaking, a technique for deducing facts about a set of interest (say in $\mathbb{R}^2$) by finding a polynomial of low degree that vanishes on this set. Our primary application will be to proving incidence theorems involving points and lines.

One particular example is as follows: Given a set P of N points in the plane, consider the set $\Delta = \{|x-y| \colon x, y \in P\}$. The *Erdös distance problem* is to determine the minimum size of the set $|\Delta|$. The problem was more or less solved recently by Guth and Katz, who followed work of Elekes and Sharir by translating the original problem into a point-line incidence problem. Guth and Katz then applied the polynomial method to solve the resulting incidence problem.

Bezout's theorem is as follows: Given two plane curves (with no common factors) of degrees d_1 and d_2, the number of points in the intersection can be at most $d_1 \cdot d_2$. An example of an application of Bezout's theorem will be to prove the following theorem of Chasles: if two collections of three lines intersect in nine points, and a cubic curve passes through eight of these points, then it must also pass through the ninth. This particular fact made an important appearance in recent work of Green and Tao on a different flavor of point-line incidence problem.

Course Outline

The outline of the course will include most of the following:

1. Algebraic tools: The polynomial ham sandwich theorem. Bezout's theorem. Cell decomposition theorem.
2. Incidence Theorems: Szemeredi-Trotter theorem. Joints problem.
3. Erdös distance problem: Translation into incidence problem. Super-duper joints theorem. A continuous analog: Falconer's problem on dimension of distance sets.
4. Other connections: Berlekamp-Welch algorithm for recoving polynomials from noisy polynomials. Thue's theorem on solutions of diophantine equations. Chasles' theorem on points lying on cubic curves.

This course will be elementary and broad. The goal is to introduce students to certain basic techniques that are being used in very recent and exciting research. We will discuss topics from algebraic geometry, combinatorics, incidence geometry, analysis, computer science, and possibly number theory. The topics are strongly motivated by recent courses taught by Larry Guth and Zeev Dvir.

Extra reading

The website for Guth's course

http://math.mit.edu/ lguth/PolynomialMethod.html

Dvir's website: look for the survey "Incidence theorems and applications" at

http://www.cs.princeton.edu/ zdvir/

Extremal and Probabilistic Combinatorics (L 16)

Béla Bollobás

For the past few decades, extremal combinatorics and probability theory have greatly influenced each other. On the one hand, probabilistic methods have permeated all branches of combinatorics, and, on the other, challenging combinatorial questions concerning isoperimetric inequalities, random graphs, disordered systems and random walks have led to new results in probability theory with a decidedly combinatorial flavour. In the course first we shall present several of the basic extremal theorems, and then we shall turn to more recent results in probabilistic combinatorics. Below are some of the topics to be covered. It is unlikely that in the course we shall do justice to all of them. Exactly which topics are examinable will be made precise during the lectures.

Pre-requisities

Students familiar with the material in Part II Probability Theory and Part II Graph Theory are bound to find the course easier.

Topics

Extremal problems concerning set systems. Sperner's Theorem, the Erdős–Ko–Rado Theorem, the Kruskal–Katona Theorem. Basic isoperimetric inequalities: vertex and edge isoperimetric inequalities in the discrete cube.

Correlation inequalities: Harris's inequality, the van den Berg–Kesten inequality, and Janson's inequality.

The discrete cube. Basic isoperimetric inequalities

Martingales. The Azuma–Hoeffding inequality, Talagrand's Inequality, and their applications.

Projections of bodies: the Box Theorem of Bollobás and Thomason. Entropy and Shearer's Inequality. Applications to hereditary properties and sumsets.

Discrete Fourier methods. The theorem of Kahn, Kalai and Linial on the influence of random variables, and the theorem of Friedgut and Kalai on sharp thresholds. Applications.

As there is no book covering these topics, the references will be to the original papers. However, to make it easier to follow the course, the lectures will be supplemented by fairly detailed printed notes. The course will be accessible to all with a smattering of probability theory and analysis.

Additive Combinatorics and Equidistribution (E16)

Non-Examinable (Graduate Level)

Péter Varjú

Additive combinatorics studies how subsets of groups, rings, etc. grow when we perform algebraic operations on them. For example, we can look at a subset A of a ring and ask whether it is possible that neither the sumset $A + A = \{x + y : x, y \in A\}$ nor the productset $A \cdot A = \{xy : x, y \in A\}$ is "significantly bigger" than A. A sum-product theorem asserts that a set A with such properties must be "close to" a subring. The meaning of the expressions between quotation marks can be defined in several ways and this leads to various sum-product theorems proved by many authors in recent years.

Additive combinatorics has found many applications in the last decade or so. Some examples are:

- the Bourgain-Glibichuk-Konyagin exponential sum estimates for multiplicative subgroups of finite fields,
- the Bourgain-Gamburd method for studying the mixing time of random walks in finite groups and in Lie groups,
- the work of Bourgain-Furman-Lindenstrauss-Mozes on stationary measures of non-commuting automorphisms of the torus.

Common features of these works are that they establish that certain probability distributions are equidistributed and the proofs use methods of additive combinatorics.

The course will discuss some of these developements and it will concentrate on applications. It will cover Bourgain's "discretized" sum-product theorem and its connection to Marstrand's projection theorem. Then random walks on $SU(2)$ will be studied.

Desirable Previous Knowledge

Some knowledge of additive combinatorics could be useful but is not necessary.

Reading to complement course material

Lecture notes will be provided at a later time.

Geometry and Topology

Algebraic Topology (M24)

Ivan Smith

This will be a first course in (co)homology theory. We will cover singular homology and cohomology, vector bundles and the Thom Isomorphism theorem, and the cohomology of manifolds up to Poincaré duality. Time permitting, there will also be some discussion of characteristic classes and cobordism, and conceivably some homotopy theory.

Desirable Previous Knowledge

We will not assume prior exposure to algebraic topology, but will move rapidly through the basics. Some knowledge of the fundamental group and of basic homology theory would be useful. We will assume familiarity with topological spaces, compactness, connectedness etc (at the level of W. Sutherland's book).

Introductory Reading

1. Sutherland, W. *Introduction to metric and topological spaces.* OUP, 1975.

Reading to complement course material

1. Hatcher, A. *Algebraic topology.* CUP, 2002.
2. Bott, R and Tu, L. *Differential forms in algebraic topology.* Springer, 1982.

Algebraic Geometry (M24)

P.M.H. Wilson

This will be a basic course introducing the tools of modern algebraic geometry, and applying them to deduce (for instance) the Riemann–Roch theorem for smooth projective curves. The most relevant reference for the course is the book of Kempf.

Topics to be covered are sheaves, abstract varieties (over an algebraically closed field) and their properties, coherent sheaves, divisors, sheaf cohomology, differentials and the Riemann–Roch Theorem. I shall not introduce schemes, but the proofs I'll give will be in such a style that there are natural extensions to the case of schemes.

Desirable Previous Knowledge

Basic theory on rings and modules will be assumed. Students will find it helpful to have looked beforehand at the book on Commutative Algebra by Atiyah and MacDonald, and/or the elementary text by Reid on Algebraic Geometry.

Introductory Reading

1. M. Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press (1988) (preliminary reading).
2. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison–Wesley (1969) (basic text also for the commutative algebra we'll need).

Reading to complement course material

1. G.R. Kempf, *Algebraic Varieties*, Cambridge University Press (1993) (main reference).
2. R. Hartshorne, *Algebraic Geometry*, Springer (1977) (more advanced text).
3. I. Shafarevich, *Basic Algebraic Geometry*, Springer (1974) (useful background reading).

3–Manifolds (M24)

Jacob Rasmussen

This will be an introduction to the topology of three-dimensional manifolds. Our understanding of this subject has advanced greatly in the last decade with the proof of Thurston's geometrization conjecture and the virtual fibering conjecture. Although these proofs are beyond the scope of the course, one of our aims will be to understand the statements of these theorems and why they are important. A second aim will be to explore the relationship between the topology of manifolds in dimensions three and four. I hope to cover the following topics:

- *Constructions.* Triangulations, handle decompositions, and Heegaard splittings. Manifolds that fiber over S^1. Dehn surgery.
- *Torsion Invariants.* Reidemeister torsion and the multivariable Alexander polynomial.
- *Embedded Surfaces.* Dehn's lemma, the Thurston norm, and Haken manifolds.
- *Geometric Structures.* Seifert fibred spaces, hyperbolic manifolds, Thurston's geometrization conjecture.
- *Relations with 4-manifolds.* Slice knots and branched double covers. Rochlin's invariant and the homology cobordism group.

There will be four examples classes.

Desirable Previous Knowledge

I will assume knowledge of the fundamental group and the classification of closed surfaces. Homology and cohomology will be used as they are developed in the Algebraic Topology lectures. Previous experience with hyperbolic geometry/metrics of constant curvature on surfaces will be useful, but is not necessary.

Introductory Reading

The first chapter of Thurston's book below.

Complementary Reading

1. J. Hempel *3-Manifolds*, PUP (1976) (AMS Reprint, 2004.)
2. W.B.R. Lickorish, *Knot Theory*, Springer (1997).
3. D. Rolfsen, *Knots and Links*, POP (1976) (AMS Reprint 2003.)
4. W. Thurston *Three-dimensional Geometry and Topology*, PUP (1997).

Differential Geometry (M24)

A. Kovalev

This course is intended as an introduction to modern differential geometry. It can be taken with a view of further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are e.g. in Analysis or Mathematical Physics. Tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, matrix Lie groups. Tangent vectors, tangent and cotangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Exterior algebra of differential forms. Orientability of manifolds. Partition of unity and integration on manifolds, Stokes' Theorem. De Rham cohomology.
- *Vector Bundles.* Structure group; principal bundles. The example of Hopf bundle. Bundle morphisms. Three views on connections: vertical and horizontal subspaces, Christoffel symbols, covariant derivative. The curvature form and second Bianchi identity.
- *Riemannian Geometry.* Connections on manifolds, torsion. Riemannian metrics, Levi–Civita connection. Geodesics, exponential map, Gauss' Lemma. Decomposition of the curvature of a Riemannian manifold, Ricci and scalar curvature, low-dimensional examples. The Hodge star and Laplace–Beltrami operator. Statement of the Hodge decomposition theorem (with a sketch-proof, time permitting).

The lectures will be supplemented by four example classes.

Printed notes will be available from `www.dpmms.cam.ac.uk/∼agk22/teaching.html`

Desirable Previous Knowledge

Essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and of multivariate calculus. The course will not assume previous knowledge of manifolds. Students might like to read some of Chapter 1 in [3] or some of [4] in advance.

References

1. R.W.R. Darling, *Differential forms and connections.* CUP, 1994.
2. S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry.* Springer-Verlag, 1990.
3. V. Guillemin, A. Pollack, *Differential topology.* Prentice-Hall Inc., 1974.
4. M. Spivak, *Calculus on manifolds.* W.A. Benjamin Inc., 1965.
5. F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, 1983.

Roughly, half of the course material is taken from [5]. The book [3] covers the required topology. On the other hand, [1], which has a chapter on vector bundles and on connections, assumes no knowledge of topology. Both [1] and [2] have a lot of worked examples. There are many other good differential geometry texts.

Symplectic Topology (L24)

Andreas Ott

Symplectic topology aims to understand the global structure of symplectic manifolds. The archetypal example of such a manifold is phase space in classical mechanics. In fact, the subject grew out of the study of Hamiltonian dynamical systems, and has over the last decades developed into an active area of research at the crossroads of dynamical systems theory, topology, algebraic geometry and theoretical physics. Among the modern techniques that proved most fruitful in detecting global properties of symplectic manifolds, pseudo-holomorphic curves as discovered by Gromov in 1985 play a prominent role. One instance of this is Gromov's celebrated non-squeezing theorem. The course will give an introduction to symplectic topology, with the goal of understanding Gromov's theorem and the main ideas of its proof.

A tentative list of topics is as follows. Review of Hamiltonian mechanics. Linear symplectic geometry. Symplectic manifolds and symplectomorphisms, Lagrangian submanifolds. Hamiltonian group actions and symplectic reduction. Complex and almost complex structures. Pseudo-holomorphic curves and their moduli space, the bubbling phenomenon and compactness. Applications of pseudo-holomorphic curves: Gromov's non-squeezing theorem, and—if time permits—an outlook on Gromov-Witten invariants and quantum cohomology.

Pre-requisite Mathematics

Some familiarity with basic notions from Differential Geometry and Algebraic Topology will be assumed. The material covered in the respective Michaelmas Term courses would be more than enough background.

Literature

1. D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, OUP (1998).
2. D. McDuff and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, AMS (1994).
3. D. McDuff and D. Salamon, *J-holomorphic Curves and Symplectic Topology*, AMS (2004).
4. A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer (2001), also available at www.math.ethz.ch/∼acannas

Complex Manifolds (L24)

J. Ross

A preliminary outline of the course is as follows, but this will almost certainly be subject to change.

- Basic concepts of complex manifolds, holomorphic vector bundles, holomorphic tangent and cotangent bundles (for which corresponding concepts from the real smooth manifolds will be assumed). Canonical line bundles, normal bundle for a submanifold and the adjunction formula.
- Brief description of sheaf cohomology, with deduction of de Rham and Dolbeault cohomology for complex manifolds.
- Hermitian metrics, connections, curvature and Chern classes for complex vector bundles. Case of holomorphic vector bundles.
- Harmonic forms: the Hodge theorem and Serre duality (general results on elliptic operators will be assumed).
- Compact Kähler manifolds. Hodge and Lefschetz decompositions on cohomology, Kodaira–Nakano vanishing, Kodaira embedding theorem.

Pre-requisite Mathematics

A knowledge of basic Differential Geometry from the Michaelmas Term course will be highly desirable. The main books for this course will be as below.

Literature

1. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978).
2. R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer (1980).
3. F. Zheng, *Complex Differential Geometry*, AMS (2000).
4. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Volume II*, Wiley (1969).
5. D. Huybrechts, *Complex Geometry*, Springer 2005.

Algebraic and arithmetic geometry (L24)

Non-Examinable (Graduate Level)

Caucher Birkar

The focus of this course is the interactions of algebraic and arithmetic geometry at an advanced level. Very roughly speaking one might define arithmetic geometry as the study of the rational solutions of polynomial equations with rational coefficients (eg, the equation $x^n + y^n + z^n = 0$). Although such equations often look quite simple but their study usually requires some of the deepest techniques developed in algebraic geometry. This connection is not really a surprise since algebraic geometry is about studying solutions of polynomial equations with coefficients in a field or even a ring.

Time permitting I am hoping to cover the basics of the following topics (with emphasis on the algebraic geometry involved): Diophantine geometry, rational points on varieties, zeta functions and L-functions, modular forms, moduli spaces of curves and abelian varieties and connections with modular forms, etc.

Pre-requisite Mathematics

Good knowledge of the foundations of algebraic geometry is required at least at the level of the part III *algebraic geometry* course (but ideally at the level of Hartshorne [H]).

Literature

[DS] F. Diamond, J. Shurman; *A first course in modular forms.* Springer.

[H] R. Hartshorne; *Algebraic geometry.* Springer.

[Hi] M. Hindry; *Introduction to zeta and L-functions from arithmetic geometry and some applications.* www.math.jussieu.fr/∼hindry/

[L] S. Lang; *Survey of Diophantine geometry.* Springer.

[P] B. Poonen; *Rational points on varieties.* math.mit.edu/∼poonen/

Irreducible holomorphic symplectic manifolds (L16)

Non-Examinable (Graduate Level)

M. Shen

Irreducible holomorphic symplectic manifolds are very interesting in the sense that they are at the crossing point of algebraic geometry, complex geometry and differential geometry. They also potentially have rich arithmetic geometry. Such manifolds are also called compact hyperkähler manifolds from the differential geometric point of view. In this course, we will study the basic geometry of these objects. Topics include Beauville–Bogomolov form, Hodge structure, Torelli type theorems and algebraic cycles. We will explain why they are higher dimensional analogue of $K3$ surfaces.

Pre-requisite Mathematics

Algebraic geometry, complex/Kähler geometry.

Literature

1. Daniel Huybrechts, Compact Hyperkähler manifolds. Invent. Math. 135(1999), No.1, 63–113.

Analytic and Birational Geometry (Easter 16)

Non-Examinable (Graduate Level)

Zhengyu Hu

This course is intended as an introduction to some problems in birational geometry of algebraic varieties over $\mathbb{C}$. Birational geometry which was initially motivated by the classification problem has made a big progress in the last decade. The focus of the course is the comparison between two approaches in this area: the algebraic method and the analytic method. The former one including the techniques from minimal model program provides a geometric picture, and leads to many results on singular normal varieties (eg.log canonical pairs). One of the crucial issues about such techniques is that, as they involve an ample divisor for the purpose of using Kawamata-Viehweg vanishing theorem, we cannot get rid of an extra positivity assumption. This becomes the main difficulty to prove a series of conjectures such as abundance conjecture. On the contrary in some special cases, the analytic method can remove the extra positivity assumption by constructing an appropriate metric from the convergence (eg. invariance of plurigenera), yet it works only on smooth varieties (or with very mild singularities). The following topics will be discussed:
1.A brief introduction of analytic geometry including currents and singular Hermitian metrics,
2.Multiplier ideals, canonical extensions and the Ohsawa-Takegoshi extension.
3.Various types of pluri-canonical extension theorems with an emphasis on a comparison between analytic and algebraic methods.

Desirable Previous Knowledge

Familiarity with basic knowledge of complex algebraic geometry such as [Sh] or parts of [H, Chapter 1-3] is required. An overview of birational geometry such as [KM,Chapter 1-3] or [Bir] is recommended.

Introductory Reading

1. [Bir] C.Birkar, Birational geometry. http://arxiv.org/abs/0706.1794.
2. [H] Hartshorne, Algebraic geometry. Springer,1977.
3. [KM] J.Kollar and S.Mori, Birational geometry of algebraic varieties.
4. [Sh] Shafarevich, Basic algebraic geometry, I and II. Springer, 1994.

Reading to complement course material

1. [Dem] Demailly, Analytic methods in algebraic geometry.
2. [Laz] Lazarsfeld, Positivity in algebraic geometry, I and II.

Logic

Category Theory (M24)

Dr Julia Goedecke

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories.

Locally small categories. The Yoneda lemma. Representations of functors.

Limits as terminal cones. Construction of limits from products and equalizers. Preservation and creation of limits.

Monomorphisms and Epimorphisms. Regular, split and strong mono- and epimorphisms.

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. The Adjoint Functor Theorems.

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions.

Abelian categories. Kernels and cokernels. Additive categories. Image factorisation in abelian categories. Exact sequences, introduction to homological algebra.

Pre-requisite Mathematics

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. Mac Lane, S. *Categories for the Working Mathematician*, Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.

2. Awodey, S. *Category Theory*, Oxford U.P. 2006. A new treatment very much in the spirit of Mac Lane's classic, but rather more gently paced.

3. Borceux, F. *Handbook of Categorical Algebra*, Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

4. McLarty, C. *Elementary Categories, Elementary Toposes* (chapters 1–12 only), Oxford U.P. 1992. A very gently-paced introduction to categorical ideas, written by a philosopher for those with little mathematical background.

Topics in Set Theory (M24)

Dr Oren Kolman

The Continuum Question is enjoying a renaissance in the 21st century. This course is a relatively self-contained introduction to independence results in contemporary set theory and their repercussions across some central fields of abstract mathematics. It focuses on the ideas and techniques in the proofs, using forcing, that the Continuum Hypothesis ($2^{\aleph_0} = \aleph_1$) can be neither proved nor refuted from the principles of ordinary set theory. We shall treat several of the following topics.

Axiomatics The formal axiomatic system of ordinary set theory (ZFC). Models of set theory. Absoluteness. Simple independence results. Transfinite recursion. Ranks. Reflection principles. Constructibility. [6]

Infinitary combinatorics Cofinality. Stationary sets. Fodor's lemma. Solovay's theorem. Cardinal exponentiation. Beth and Gimel functions. Generalized Continuum Hypothesis. Singular Cardinals Hypothesis. Prediction principles (diamonds, squares, black boxes). Partial orders. Aronszajn and Suslin trees. Martin's Axiom. Suslin's Hypothesis. [6]

Forcing Generic extensions. The forcing theorems. Examples. Adding reals; collapsing cardinals. Introduction to iterated forcing. Internal forcing axioms. Proper forcing. [6]

Large cardinals Introduction to large cardinals. Ultrapowers. Scott's theorem. [2]

Partition relations and possible cofinality theory Partition relations. Model–theoretic methods. Ramsey's theorem; Erdős–Rado theorem. Kunen's theorem. Walks on ordinals. Todorcevic's theorem. Introduction to pcf theory. [4]

Pre-requisite Mathematics

The Part II course *Logic and Set Theory* or its equivalent is essential.

Literature

Basic material
† Drake, F. R., Singh, D., *Intermediate Set Theory*, John Wiley, Chichester, 1996.
Eklof, P. C., Mekler, A. H., *Almost Free Modules*, rev. ed., North-Holland, Amsterdam, 2002.
Halbeisen, L., *Combinatorial Set Theory With a Gentle Introduction to Forcing*, Springer, Berlin, 2012.
Kanamori, A., *The Higher Infinite*, 2^{nd} ed., Springer, Berlin, 2009.
† Kunen, K., *Set Theory*, reprint, Studies in Logic, 34, College Publications, London, 2011.

Advanced topics
Burke, M. R., Magidor, M., Shelah's pcf theory and its applications, Ann. Pure Appl. Logic 50 (1990), 207–254.
Kanamori, A., Foreman, M., *Handbook of Set Theory*, Springer, Berlin, 2012.
† Shelah, S., *Proper and Improper Forcing*, 2^{nd} ed., Springer, Berlin, 1998. Chapters 1 and 2.
Shelah, S. *Cardinal Arithmetic*, Oxford University Press, New York, 1994.
Todorcevic, S., Combinatorial dichotomies in set theory, Bull. Symbolic Logic 17 (2011), 1–72.

Topos Theory (L24)

Prof. P.T. Johnstone

The class of categories known as toposes were first introduced by Alexander Grothendieck in the early 1960s, in order to provide a mathematical underpinning for the 'exotic' cohomology theories needed in algebraic geometry. Every topological space X gives rise to a topos (the category of sheaves of sets on X) and every topos in Grothendieck's sense can be considered as a 'generalized space'.

At the end of the same decade, William Lawvere and Myles Tierney realized that Grothendieck's notion of topos could, if reformulated in an elementary manner, become a categorical foundation for the whole of mathematics, by providing an abstract notion of a 'universe of sets' within which one could carry out most familiar set-theoretic constructions, but which also, thanks to the geometrical examples developed by Grothendieck, provides one with much more freedom than classical (ZF-style) set theory to construct 'new worlds from old' having particular properties.

The ensuing development of topos theory opened up a rich seam of potential interaction between geometry and logic, which is still being mined for new and interesting results. The course will begin by developing the basic theory of toposes and geometric morphisms, with the aim of reaching the theory of classifying toposes, and the result that every topos in the sense of Grothendieck is the classifying topos of a suitable first-order theory, by the end of the term.

Pre-requisite Mathematics

Knowledge of the material of the Michaelmas Term course on Category Theory is essential. Some familiarity with classical first-order logic (such as is provided by the Part II course on Logic and Set Theory) would be very desirable but not essential. No previous knowledge of sheaf theory is required.

Literature

1. McLarty, C.: *Elementary Categories, Elementary Toposes*, Oxford U.P. 1992. Recommended for preliminary reading; it's a very gently-paced introduction to the subject, written for philosophers with little mathematical background.

2. Mac Lane, S. and Moerdijk, I.: *Sheaves in Geometry and Logic: a First Introduction to Topos Theory*, Springer-Verlag 1992. The best available textbook on the subject, though its approach diverges in several respects from that which will be adopted in the course.

3. Johnstone, P.T.: *Sketches of an Elephant: a Topos Theory Compendium*, Oxford U.P. (2 volumes), 2002. Emphatically not a textbook, but the best place to find out anything you want to know about toposes.

Computability and Logic (L24)

Thomas Forster

This course is conceived as the sequel to Part II Logic and Set Theory. It is less general than that course, partly because its point of departure is a determination to fill a *lacuna* in Part II, namely the theory of computable functions. The dual nature of the title reflects the desire of the lecturer to decorate a treatment of recursive function theory with as many bits of logic as can be sensibly fitted in to such a narrative.

Structural induction and wellfounded induction. Primitive recursive functions, General recursive functions. Ackermann's function. Goodstein's function. Finite state machines. Gödel's β-function. Turing Machines. Decidable and semidecidable sets. Existence of a Universal Turing machine. Kleene's T-function. Unsolvability of the Halting Problem. Fixed-point theorem. Rice's theorem. Immune and productive sets, recursive inseparability. Hierarchies of fast-growing functions. Trakhtenbrot's theorem.

Gödel's incompleteness theorem. Recursive ordinals. Representation of computable functions by λ-terms. Kleene-Post. Friedberg-Muchnik and the Priority method. Baker-Gill-Solovay. Automatic theories. Automatic structures. Recursive structures, Tennenbaum's theorem.

Pre-requisite Mathematics

I shall assume that everybody has done Part II Set Theory and Logic, though this is actually overkill.

Literature

There are numerous good books on this subject. The following is in paperback, and holders of a Cambridge Blue card can acquire it from the CUP bookshop in town for a 15% discount.

1. Cutland, N. Computability, Cambridge University Press

There is a wealth of material available in links from the lecturer's home page: www.dpmms.cam.ac.uk/~tf.

Number Theory

Elliptic Curves (M24)

T.A. Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points. The first few lectures will include a review of the necessary geometric background (at the level of Chapters I and II of [2]). The following topics will be covered, and possibly others if time is available. There will be examples sheets and examples classes.

Weierstrass equations and the group law. Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process. Associativity via the identification with the Jacobian. Elliptic curves as group varieties.

Isogenies. Definition and examples. The degree of an isogeny is a quadratic form. The invariant differential and separability. Description of the torsion subgroup over an algebraically closed field.

Elliptic curves over finite fields. Hasse's theorem.

Elliptic curves over local fields. Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.

Elliptic curves over number fields. The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Descent by 2-isogeny. Numerical examples.

Desirable Previous Knowledge

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. It would also be useful to have some rudimentary knowledge of algebraic curves and of the field of p-adic numbers.

Introductory Reading

1. J.H. Silverman, J. Tate, Rational Points on Elliptic Curves, Springer, 1992.

Reading to complement course material

1. J.W.S. Cassels, Lectures on Elliptic Curves, CUP, 1991.
2. J.H. Silverman, The Arithmetic of Elliptic Curves, Springer, 1986.

Modular Forms (M24)

James Newton

This course will be an introduction to the theory of modular forms. Modular forms are special holomorphic functions on the complex upper half plane. Despite their apparently analytic definition, these objects encode arithmetic information (they have Fourier expansions whose coefficients are often arithmetically interesting), and are closely related to the geometry of Riemann surfaces/algebraic curves obtained as quotients of the upper half plane by discrete group actions ('modular curves').

Modular forms have numerous applications in number theory, as well as being interesting objects of study in their own right (particularly when viewed as examples of automorphic representations). Their most famous appearance is in the proof of Fermat's last theorem, but there are more elementary applications to arithmetic identities and congruences and the theory of quadratic forms. There are also other more advanced applications to the class number one and congruent number problems.

Topics to be covered include:

1. Basic definitions and examples, Eisenstein series, theta series
2. Modular forms of level one, congruence subgroups and modular forms of higher level
3. Modular curves as Riemann surfaces, construction and application to dimension formulae for spaces of modular forms
4. Hecke operators and Hecke eigenforms
5. L-functions of modular forms, analytic continuation and functional equation

If time permits, additional topics such as congruences between modular forms or special values of modular functions will be discussed. Examples classes will be offered to support the lectures.

Desirable Previous Knowledge

A working knowledge of complex analysis (including, for example, Cauchy's residue theorem) will be required, and some familiarity with Riemann surfaces and/or algebraic curves is desirable (for example, we will make use of the Riemann–Hurwitz formula and Riemann–Roch theorem).

Introductory Reading

1. Chapter VII of J.-P. Serre, A course in arithmetic, Springer, 1973 (or the French original, Cours d'arithmétique, Presses Universitaires de France, 1970).
2. D. Zagier, Elliptic modular forms and their applications, in The 1-2-3 of modular forms, Springer, 2008.

Reading to complement course material

1. F. Diamond and J. Shurman, A first course in modular forms, Springer, 2005.
2. J. S. Milne, Modular functions and modular forms, online lecture notes available from `http://www.jmilne.org/math/CourseNotes/mf.html`.
3. W. Stein, Modular forms, a computational approach, AMS, 2007. Also available online at `http://wstein.org/books/modform/modform/`.

Algebraic Number Theory (L24)

A J Scholl

In recent years one of the most growing areas of research in number theory has been Arithmetic Algebraic Geometry, in which the techniques of algebraic number theory and abstract algebraic geometry are applied to solve a wide range of deep number-theoretic problems. These include the celebrated proof of Fermat's Last Theorem, the Birch–Swinnerton-Dyer conjectures, the Langlands Programme and the study of special values of L-functions. In this course we will study one half of the picture: Algebraic Number Theory. I will assume some familiarity with the basic ideas of number fields, although these will be reviewed briefly at the beginning of the course. (The relevant algebra will also be found in the Commutative Algebra course.)

Topics likely to be covered:

Decomposition of primes in extensions, decomposition and inertia groups. Discriminant and different.

Completion, adeles and ideles, the idele class group. Application to class group and units.

Dedekind zeta function, analytic class number formula.

Class field theory (statements and applications). L–functions.

Pre-requisite Mathematics

A first course in number fields (or equivalent reading). Basic algebra up to and including Galois theory is essential.

Literature

1. J.W.S. Cassels and A. Fröhlich, Algebraic Number Theory. London Mathematical Society 2010 (2nd ed.)

2. A, Fröhlich, M.J. Taylor, Algebraic Number Theory. Cambridge, 1993.

3. J. Neukirch, Algebraic number theory. Springer, 1999.

The Riemann Zeta Function (L24)

Adam Harper

The Riemann zeta function $\zeta(s)$ is our most important tool for studying the distribution of prime numbers. It was introduced by Riemann in 1859, building on work of Euler, and remains the subject of huge amounts of current research. This course will introduce the zeta function and explore what is known about its zeros, some other aspects of its behaviour, and applications of this information.

The course will cover some of the following topics, depending on time and audience preferences:

1. *Basic theory.* Definition of $\zeta(s)$ when $\Re(s) > 1$, and then when $\Re(s) > 0$ and for all s. The connection with primes via the Euler product. Proof that $\zeta(s) \neq 0$ when $\Re(s) \geq 1$, and deduction of the Prime Number Theorem. The explicit formula for counting primes.

2. *Zero-free regions.* Non-existence of zeta zeros follows from estimates for $\sum_{N<n<2N} n^{it}$. The connection with exponential sums, and the methods of Van der Corput and Vinogradov. Zero-free regions for $\zeta(s)$. Application to improving the Prime Number Theorem. Statement of the Riemann Hypothesis.

3. *Zero-density estimates.* Use of Dirichlet polynomials to detect zeros. Mean and large values estimates for Dirichlet polynomials, and consequences for the density of zeta zeros. Application to counting primes in short intervals. Statement of the Density Hypothesis.

4. *The zeta function when* $\Re(s) = 1/2$. The moments $\int_0^T |\zeta(1/2+it)|^{2k} dt$ of the zeta function, and the connection with random matrix theory. Selberg's central limit theorem for $\log \zeta(1/2+it)$. The pair correlation of zeta zeros.

Pre-requisite Mathematics

There are no pre-requisites beyond basic real and complex analysis (up to Cauchy's Residue Theorem). The course will have a flavour of estimating complicated objects and handling error terms, which might be familiar from previous courses in analysis or probability.

Literature

A. Ivić. *The Riemann Zeta-Function. Theory and Applications.* Dover edition, published by Dover Publications, Inc.. 2003

E. C. Titchmarsh. *The Theory of the Riemann Zeta-function.* Second edition, revised by D. R. Heath-Brown, published by Oxford University Press. 1986

Titchmarsh's book is the classic text on the zeta function. It has nice introductory chapters and also covers most of the material on zero-free regions. Ivić's book is more advanced, but covers most of the material in the course as well as lots more. Standard texts on analytic number theory, such as Davenport, *Multiplicative Number Theory*; Iwaniec and Kowalski, *Analytic Number Theory*; Montgomery and Vaughan, *Multiplicative Number Theory*; will also cover some of the material and provide nice background. (Iwaniec and Kowalski's book is particularly comprehensive, though rather advanced.)

Probability

Advanced Probability. (M24)

A. Sola and P. Sousi

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

As a complement to the lectures, example sheets will be handed out, and supervisions will be given.

The main topics are as follows:

Review of measure and integration: sigma-algebras, measures, and filtrations; integrals and expectation; Fatou's lemma, monotone and dominated convergence; product measures, independence, and Fubini's theorem.

Conditional expectation: Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.

Martingales: Discrete time martingales, submartingales and supermartingales; optional stopping; Doob's inequalities, upcrossings, convergence theorems, backwards martingales. Applications.

Weak convergence: Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.

Sums of independent random variables: Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.

Stochastic processes in continuous time: Kolmogorov's criterion, regularization of paths; martingales in continuous time.

Brownian motion: Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; the Dirichlet problem and Brownian motion; Donsker's theorem.

Poisson random measures: Definitions, compound Poisson processes; Infinite divisibility, the Lévy-Khinchin formula, Lévy-Itô decomposition.

Desirable Previous Knowledge

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Introductory Reading

1. D. Williams, Probability with martingales, Cambridge University Press (CUP), 1991.

Reading to complement course material

1. D. Applebaum, Lévy processes (2nd ed.), CUP 2009.
2. R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
3. O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.

Concentration in Discrete Random Processes (M16)

Non-Examinable (Graduate Level)

Lutz Warnke

In probabilistic combinatorics the concentration of measure phenomenon is of great importance. For example, in the theory of random graphs, we are often interested in showing that certain random variables are concentrated around their expected values. In this course we study a variation of this theme, where we are interested in proving that certain random variables remain concentrated around their expected trajectories as an underlying random process evolves.

In particular, we focus on Wormald's *differential equation method*, which allows for proving such dynamic concentration results. This method systematically relates the evolution of a given random process with an associated system of differential equations, and the basic idea is that the solution of the differential equations can be used to approximate the dynamics of the random process.

We begin by discussing the basic method and several examples. Afterwards we introduce an extension due to Bohman, which allows for a much wider range of applications. Finally, if time permits, we also briefly discuss 'self-correction', which allows for very tight dynamic concentration results.

Desirable Previous Knowledge

We shall only assume some basic notions of probability and graph theory.

Literature

The following articles contain a substantial proportion of the material we intend to cover (and much more).

1. N. Wormald, Differential equations for random processes and random graphs, *The Annals of Applied Probability* **5** (1995), 1217–1235.
2. N. Wormald, The differential equation method for random graph processes and greedy algorithms, in *Lectures on Approximation and Randomized Algorithms*, PWN, Warsaw, (1999), 73–155
3. T. Bohman, The triangle-free process, *Advances in Mathematics* **221** (2009) 1653-1677.
4. T. Bohman, A. Frieze,E. Lubetzky, A note on the random greedy triangle-packing algorithm, *Journal of Combinatorics* **1** (2010), 477–488.

Discrete complex analysis and conformal invariance (M8)

Non-Examinable (Graduate Level)

Zhongyang Li

The course is about the conformal invariance exhibited in lattice models in statistical mechanics, focusing on the dimer model (perfect matching) and the Ising model.

Conformal invariance of a lattice-based statistical mechanical system is a symmetry property of the system at large scales. It says that, in the limit as the lattice spacing tends to zero, macroscopic quantities associated with the system transform covariantly under the conformal maps of the domain. Conformal invariance is an extremely powerful principle, which have been used by physicists fruitfully to compute exact critical exponents and other physical quantities associated to critical lattice models. Although many well-known models are believed to be conformally invariant for a long time, only recently mathematicians are able to prove it rigorously.

The fundamental technique to study the conformal invariance is discrete complex analysis. In this course, we will talk about discretizations of harmonic and holomorphic functions and discuss discrete analogues of the usual complex analysis theorems. By considering the approximations of the discrete objects to their continuous counterparts, we will talk about applying the technique of discrete complex analysis to show the conformal invariance of dimer and Ising models at their critical point. The topics of universality and isoradial graphs will also be covered.

This course may also be interesting to students intending to take the course Schramm-Loewner Evolutions in Lent Term, as it introduces some of the techniques used to prove convergence to SLE for certain discrete systems. There will be essentially no overlap in content of the two courses.

Pre-requisite Mathematics

There are no essential pre-requisites beyond probability, and complex analysis at undergraduate levels.

Literature

1. R. Kenyon, Conformal invariance of domino tiling, Ann. Probab. 28 (2000), no. 2, 759-795
2. R. Kenyon, The laplacian and $\overline{\partial}$ operators on critical planar graphs, Invent. Math. 150 (2002), no. 2, 409-439.
3. D. Chelkak, S. Smirnov, Discrete complex analysis on isoradial graphs, Adv. Math. 228 (2011), no. 3, 1590-1630
4. D. Chelkak, S. Smirnov, Universality in the 2D Ising model and conformal invariance of fermionic observables, Invent. Math. 189 (2012), no. 3, 515-580

Number of lectures

8, non-examinable, graduate level.

Stochastic Calculus and Applications (L24)

Michael Tehranchi

This course is an introduction to the theory of continuous-time stochastic processes, with an emphasis on the central role played by Brownian motion. It complements the material in Advanced Probability, Advanced Financial Models, and Schramm–Loewner Evolutions.

- *Review of Brownian motion.* Isonormal process. Wiener's existence theorem. Sample path properties.
- *Continuous stochastic calculus.* Martingales, local martingales and semi-martingales. Quadratic variation and co-variation. Itô's isometry and definition of stochastic integral. Kunita–Watanabe's theorem. Itô's formula.
- *Applications to Brownian motion.* Lévy's characterization of Brownian motion. Dubins–Schwartz theorem. Girsanov's theorem. Transience and recurrence. Martingale representation theorems.
- *Stochastic differential equations.* Strong and weak solutions. Notions of existence and uniqueness. Yamada–Watanabe theorem. Strong Markov property. Kolmogorov, Fokker–Planck and Feynmann–Kac partial differential equations. The one-dimensional case. Stochastic partial differential equations.

Pre-requisite Mathematics

Knowledge of measure theoretic probability at the level of Part III Advanced Probability will be assumed, especially familiarity with discrete-time martingales and basic properties of Brownian motion.

Literature

1. I. Karatzas and S. Shreve. (1998) Brownian Motion and Stochastic Calculus. Springer.
2. D. Revuz and M. Yor. (2001) Continuous martingales and Brownian motion. Springer.
3. L.C. Rogers and D. Williams. (2002) Diffusions, Markov Processes and Martingales. Vol.1 and 2. Cambridge University Press.

Percolation and Related Topics (L16)

Geoffrey Grimmett and Demeter Kiss

The percolation process is the simplest probabilistic model for a random medium in finite-dimensional space. It has a central role in the general theory of disordered systems arising in the mathematical sciences, and it has strong connections with statistical mechanics. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to the theory of phase transitions in physics.

The basic theory of percolation will be described in this course, with some emphasis on areas for future development. The fundamental techniques, including correlation and/or concentration inequalities and their ramifications, will be covered. The related topics may include self-avoiding walks, and further models from interacting particle systems, and (if time permits) certain physical models for the ferromagnet such as the Ising and Potts models.

Pre-requisite Mathematics

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

The following texts will cover the majority of the course, and are available online.

Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2010;
see `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`

Grimmett, G. R., *Three theorems in discrete random geometry*, Probability Surveys 8 (2011) 403–411, `http://arxiv.org/abs/1110.2395`

Schramm-Loewner Evolutions (L16)

L. Dumaz & J. R. Norris

Schramm-Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for $\kappa \leq 4$ the curves are simple, for $4 \leq \kappa < 8$ they have double points but are non-crossing, and for $\kappa \geq 8$ they are space-filling. We will then explore the properties of the curves for a number of special values of κ (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

Literature

1. Nathanaël Berestycki and James Norris. Lecture notes on SLE.
   `http://www.statslab.cam.ac.uk/~james/Lectures`

2. Wendelin Werner. *Random planar curves and Schramm-Loewner evolutions*, arXiv:math.PR/0303354, 2003.

3. Gregory F. Lawler. *Conformally Invariant Processes in the Plane*, AMS, 2005.

Statistics

Statistical Theory (M16)

Richard Nickl

The course gives a mathematical introduction to same selected core topics of statistical theory. It is complemented by the Lent term course on 'Nonparametric Statistical Theory'.

We will start with a review of the main ideas and with a rigorous account of classical parametric statistical models. The theory of consistency and asymptotic normality is developed in the general setting of 'M-estimators' (including maximum likelihood estimators and nonlinear least squares procedures). The results are imbedded in the unifying framework of locally asymptotically normal statistical models (a notion due to Le Cam). Moreover the Bayesian perspective on general parametric models will be discussed, and some corresponding theory will be developed.

The course then proceeds to extend the standard Gaussian linear model to the setting where the number p of parameters is possibly larger than the sample size n. This situation, which is important in many applications and has received a good deal of attention recently, requires new methods that go beyond classical least squares procedures. The main ideas of ℓ_1-minimisation procedures (LASSO) will be laid out, showing that one can estimate low-dimensional models that are embedded in very high-dimensional structures with almost no loss of accuracy compared to the classical situation where the 'position' of the model is known. As a high point we will prove a key result from *compressed sensing*, namely that high-dimensional Gaussian sensing matrices satisfy the so-called *restricted isometry property.*

Pre-requisite Mathematics

Basic courses in statistics and probability are required. Having taken advanced courses in both subjects will clearly be helpful but is not necessary and students who have found their first statistics courses boring are strongly encouraged to attend.

Literature

Complete lecture notes for the first part of the course are already available on the lecturers website (only Chapter 2 in these notes is relevant). Time (and skill) of the lecturer permitting lecture notes for the second part will be made available too. For those who want to read ahead on the second part of the course the recent book by P. Bühlmann and S. van de Geer, *Statistics for high-dimensional data*, Springer 2011, can be recommended.

Actuarial Statistics (M16)

S.M. Pitts

This course provides an introduction to various topics in non-life insurance mathematics. These topics feature mainly in the Institute and Faculty of Actuaries examination CT6.

Topics covered in lectures include

1. Loss distributions
2. Reinsurance
3. Aggregate claims
4. Ruin theory
5. Credibility theory
6. No claims discount systems

Pre-requisite Mathematics

This course assumes

an introductory probability course (including moment generating functions, probability generating functions, conditional expectations and variances)

a statistics course (including maximum likelihood estimation, Bayesian methods)

that you know what a Poisson process is

that you have met discrete time finite statespace Markov chains

Literature

1. S. Asmussen and H. Albrecher *Ruin Probabilities.* 2nd edition. World Scientific, 2010.
2. C.D. Daykin, T. Pentikäinen and E. Pesonen, *Practical Risk Theory for Actuaries and Insurers.* Chapman and Hall, 1993.
3. D.M. Dickson, *Insurance Risk and Ruin.* CUP, 2005.
4. J. Grandell, *Aspects of Risk Theory.* Springer, 1991.
5. R.J. Gray and S.M. Pitts, *Risk Modelling in General Insurance: From Principles to Practice.* CUP, 2012.
6. T. Rolski, H. Schmidli, V. Schmidt and J. Teugels, *Stochastic Processes for Insurance and Finance.* Wiley, 1999.

Biostatistics (M10+L14)

This course consists of two components: Survival Data and Statistics in Medical Practice. Together these make up one 3 unit (24 lecture) course. You must take both components together for the examination. Survival Data has 14 lectures; Statistics in Medical Practice has 10 lectures.

Statistics in Medical Practice (M10)

R. Turner, C. Jackson, J. Wason, J. Bowden, D. de Angelis, S. Seaman

Each lecture will be a self-contained study of a topic in biostatistics, which may include clinical trials, meta-analysis, missing data, multi-state models and infectious disease modelling. The relationship between the medical issue and the appropriate statistical theory will be illustrated.

Pre-requisite Mathematics

Undergraduate-level statistical theory, including estimation, hypothesis testing and interpretation of findings.

Literature

There are no course books, but relevant medical papers will be made available before some lectures for prior reading. It would be very useful to have some familiarity with media coverage of medical stories involving statistical issues, e.g. from Behind the Headlines on the NHS Choices website: http://www.nhs.uk/News/Pages/NewsIndex.aspx

Reading to complement course material

1. Armitage P, Berry G, Matthews JNS. Statistical Methods in Medical Research. Wiley-Blackwell, 2001.
2. Borenstein M, Hedges L, Higgins JPT, Rothstein HR. Introduction to Meta-Analysis. Wiley, 2009.
3. Jennison C, Turnbull B. Group Sequential Methods with Applications to Clinical Trials. Chapman and Hall, 2000.

Survival Data (L14)

Dr P. Treasure

Fundamentals of Survival Analysis:
Characteristics of survival data; censoring. Definition and properties of the survival function, hazard and integrated hazard. Examples.

Review of inference using likelihood. Estimation of survival function and hazard both parametrically and non-parametrically.

Explanatory variables: accelerated life and proportional hazards models. Special case of two groups. Model checking using residuals.

Current Topics in Survival Analysis:
In recent years there have been lectures on: frailty, cure, relative survival, empirical likelihood, counting processes and multiple events.

Pre-requisite Mathematics

Literature

1. D. R. Cox and D. Oakes, Analysis of Survival Data, London: Chapman and Hall (1984).
2. P. Armitage, J. N. S. Matthews and G. Berry, Statistical Methods in Medical Research (4th ed.), Oxford: Blackwell (2001) [Chapter on Survival Analysis for preliminary reading].
3. M. K. B. Parmar and D. Machin, Survival Analysis: A Practical Approach (1995), Chichester: John Wiley.

Applied Statistics (Michaelmas and Lent (24))

Susan Pitts, Jenny Wadsworth and Brian Tom

This is a three unit course, with 16 hours (8 lectures and 8 classes) in the Michaelmas Term and 8 hours (4 lectures and 4 classes) in the Lent Term. It is a *practical* course aiming to develop skills in analysis and interpretation of data. Students are strongly encouraged to attend the course Statistical Theory for the theoretical background to the results used in the practical analysis of data.

The statistical methods listed below will be put into practice using R. In the practical classes, emphasis is placed on the importance of the clear presentation of the analysis, so that students are given the opportunity to submit some written solutions to the lecturer.

Syllabus

Michaelmas Term

Introduction to R. Use of LaTeXfor report writing. Exploratory data analysis, graphical summaries.

Linear regression and its assumptions. Relevant diagnostics: residuals, Q-Q plots, leverages, Cook's distances and related methods. Hypothesis tests for linear models, ANOVA, F-tests. Interpretation of interactions.

The essentials of generalised linear modelling. Discrete data analysis: binomial and Poisson regression. Diagnostics, goodness-of-fit and model selection.

Lent Term

Some special topics. Previous examples include generalised additive models, and longitudinal data analysis.

Pre-requisite Mathematics

It is assume that you will have done an introductory statistics course, including: elementary probability theory; maximum likelihood; hypothesis tests (t-tests, χ^2-tests, F-tests); confidence intervals.

Literature

(a) Dobson, A.J. and Barnett A. (2008) *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC.

(b) Agresti, A. (1990) *Categorical Data Analysis.* Wiley.

(c) McCullagh, P. and Nelder, J.A. (1989) *Generalized Linear Models.* Chapman & Hall.

(d) Venables, W.N. and Ripley, B.D. (2002) *Modern Applied Statistics with S.* Springer-Verlag. 4th edition.

(e) Pawitan, Y. (2001) *In All Likelihood: Statistical Inference Using Likelihood.* Oxford Science Publications.

Nonparametric Statistical Theory (L16)

Richard Samworth and Arlene Kim

In parametric Statistics, it is assumed the data comes from a known finite-dimensional family of distributions. While that assumption is often convenient, it may not always be true; in this course, we will ask whether it is possible to construct procedures which do not rely on such assumptions. We will see that, in many cases, the standard maximum likelihood approach fails, and we must instead use procedures designed specically for nonparametric settings.

We will focus on fundamental problems, such as estimating a distribution function, density, or regression function, and describe techniques including empirical distribution functions, kernels and splines. We will see that much progress can be made, although several open problems remain.

Desirable Previous Knowledge

Basic knowledge of Statistics, probability and analysis is required. Measure theory is not required, but would be a small bonus. This course complements the Michaelmas term course on Statistical Theory.

Introductory Reading

(a) Wand, M. P. and Jones, M. C. (1995) Kernel Smoothing. Chapman and Hall.

(b) Wasserman, L. (2006) All of Nonparametric Statistics. Springer.

Reading to complement course material

(a) Fan, J. and Gijbels, I. (1996) Local Polynomial Modelling and Its Applications. Chapman and Hall.

(b) Green, P. J. and Silverman, B. W. (1994) Nonparametric regression and Generalized Linear Models. Chapman and Hall.

(c) van der Vaart, A. W. (1998) Asymptotic Statistics. Cambridge.

Applied Bayesian Statistics (L11+5)

David Spiegelhalter

This course will count as a 2-unit (16 lecture) course. There will be 11 lectures and five practical classes.

- Bayes theorem; principles of Bayesian reasoning; probability as a subjective construct
- Exact conjugate analysis; exponential family; mixture priors
- Likelihood principle; alternative theories of inference
- Assessment of prior distributions; imaginary observations
- Monte Carlo analysis;
- Conditional independence; graphical models
- Markov chain Monte Carlo methods; convergence
- Regression analysis (linear, GLM, nonlinear)
- Model criticism and comparison; Bayesian P-values; information criteria
- Hierarchical models (GLMMs)

The practical classes will use WinBUGS.

Pre-requisite Mathematics

This course assumes that students have a working knowledge of non-Bayesian applied statistics, such as the Applied Statistics course. It will be helpful but not essential to attend the Monte Carlo Inference course. Full familiarity with properties and manipulations of probability distributions will be assumed, including marginalisation, change of variable, Fisher information, iterated expectation, conditional independence, and so on.

Literature

(a) Lunn, D., Jackson, C., Best, N.G., Thomas, A. and Spiegelhalter, D.J. (2012) *The BUGS Book: A Practical Introduction to Bayesian Analysis.* Chapman and Hall.

(b) Gelman A., Carlin, J.B., Stern, H.S., and Rubin, D.B. (2003) *Bayesian Data Analysis.* 2nd Edition.Chapman and Hall.

Time Series and Monte Carlo Inference (L16)

The course consists of two components: Time Series and Monte Carlo Inference, each having 8 lectures. Together these make up one 2 unit (16 lecture) course. You must take the two components together for the examination.

Time Series (L8)

Yi Yu

Time series analysis refers to problems in which observations are collected at regular time intervals and there are correlations among successive observations. Applications cover virtually all areas of Statistics but some of the most important include economic and financial time series, and many areas of environmental or ecological data. This course will cover some of the most important methods for dealing with these problems, including basic definitions of autocorrelations etc., time-domain model fitting including autoregressive and moving average processes, and spectral methods.

Pre-requisite Mathematics

You should have attended introductory Probability and Statistics courses.

Literature

(a) P. J. Brockwell and R. A. Davis, Time Series: Theory and Methods. Springer Series in Statistics (2006).

(b) C. Chatfield, The Analysis of Time Series: Theory and Practice. Chapman and Hall (2004).

(c) W. A. Fuller, Introduction to statistical time series. Vol. 428. Wiley-Interscience (2009).

(d) G. E. P. Box, G. M. Jenkins and G. C. Reinsel. Time series analysis: forecasting and control. Vol. 734. Wiley (2011).

Monte Carlo Inference (L8)

Alexandra Carpentier

Monte Carlo methods are concerned with the use of stochastic simulation techniques for statistical inference. These have had an enormous impact on statistical practice, especially Bayesian computation, over the last 20 years, due to the advent of modern computing architectures and programming languages. This course covers the theory underlying some of these methods and illustrates how they can be implemented and applied in practice.

The following topics will be covered: Techniques of random variable generation. Markov chain Monte Carlo (MCMC) methods for Bayesian inference. Gibbs sampling, Metropolis-Hastings algorithm, reversible jump MCMC.

Pre-requisite Mathematics

You should have attended introductory Probability and Statistics courses. A basic knowledge of Markov chains would be helpful. Prior familiarity with a statistical programming package such as R or MATLAB would also be useful.

Literature

(a) P. J. E. Gentle, Random Number Generation and Monte Carlo Methods (Second Edition). Springer (2003).

(b) B. D. Ripley, Stochastic Simulation. Wiley (1987).

(c) W.D. Gamerman and H. F. Lopes, Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference (Second Edition). Chapman and Hall (2006).

(d) C.P. Robert and G. Casella, Monte Carlo Statistical Methods. Springer (1999).

Operational Research and Mathematical Finance

Mathematics of Operational Research (M24)

F. Fischer

This course is accessible to a candidate with mathematical maturity who has no previous experience of operational research; however it is expected that most candidates will already have had exposure to some of the topics listed below.

- Lagrangian sufficiency theorem. Lagrange duality. Supporting hyperplane theorem. Sufficient conditions for convexity of the optimal value function. Fundamentals of linear programming. Linear program duality. Shadow prices. Complementary slackness. [2]
- Simplex algorithm. Two-phase method. Dual simplex algorithm. Gomory's cutting plane method. [3]
- Complexity of algorithms. NP-completeness. Exponential complexity of the simplex algorithm. Polynomial time algorithms for linear programming. [2]
- Network simplex algorithm. Transportation and assignment problems, Ford-Fulkerson algorithm, max-flow/min-cut theorem. Shortest paths, Bellman-Ford algorithm, Dijkstra's algorithm. Minimum spanning trees, Prim's algorithm. MAX CUT, semidefinite programming, interior point methods. [5]
- Branch and bound. Dakin's method. Exact, approximate, and heuristic methods for the travelling salesman problem. [3]
- Cooperative and non-cooperative games. Two-player zero-sum games. Existence and computation of Nash equilibria, Lemke-Howson algorithm. Bargaining. Coalitional games, core, nucleolus, Shapley value. Mechanism design, Arrow's theorem, Gibbard-Satterthwaite theorem, VCG mechanisms. Auctions, revenue equivalence, optimal auctions. [9]

Books

(a) M.S. Bazaraa, J.J. Jarvis and H.D. Sherali: Linear Programming and Network Flows, Wiley (1988).

(b) D. Bertsimas, J.N. Tsitsiklis. Introduction to Linear Optimization. Athena Scientific (1997).

(c) N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani. Algorithmic Game Theory. Cambridge University Press (2007).

(d) M. Osborne, A. Rubinstein: A Course in Game Theory. MIT Press (1994).

Advanced Financial Models (M24)

Michael Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability, Stochastic Calculus and Applications, and Optimal Investment.

- *Discrete time models.* Filtrations and martingales. Arbitrage, state price densities and equivalent martingale measures. Attainable claims and market completeness. European and American claims. Optimal stopping.
- *Brownian motion and stochastic calculus.* Brief survey of stochastic integration. Girsanov's theorem. Itô's formula. Martingale representation theorem.

- *Continuous time models.* Admissible strategies. Black–Scholes model. The implied volatility surface. Pricing and hedging via partial differential equations. Dupire's formula. Stochastic volatility models.
- *Interest rate models.* Short rates, forward rates and bond prices. Markovian short rate models. The Heath–Jarrow–Morton drift condition.

Pre-requisite Mathematics

A knowledge of probability theory at the level of Part II Probability and Measure will be assumed. Familiarity with Part II Stochastic Financial Models is helpful.

Literature

Lecture notes will be distributed. Additionally, the following books may be helpful.

(a) M. Baxter & A. Rennie. (1996) Financial calculus: an introduction to derivative pricing. Cambridge University Press

(b) M. Musiela and M. Rutkowski. (2006) Martingale Methods in Financial Modelling. Springer.

(c) D. Kennedy. (2010) Stochasic Financial models. Chapman & Hall

(d) Lamberton, D. & B. Lapeyre. (1996) Introduction to stochastic calculus applied to finance. Chapman & Hall

(e) S. Shreve. (2005) Stochastic Calculus for Finance: Vol. 1 and 2. Springer-Finance

Designing Online Contests (L24)

Milan Vojnović

This course uses game theory to provide theoretical underpinnings for the design of contests that arise in various real-life situations. A particular focus is devoted to the elements of contests that arise in the context of the design of Internet e-commerce and online services. Broadly speaking, a contest is a system in which agents invest efforts in order to win one or more prizes. The goal of a contest owner is to maximize a given objective by using a suitable prize allocation mechanism; for example, the objective may be the total contribution solicited from the contestants. The course focuses on the analysis of strategic equilibrium including the level of total contribution, maximum individual contribution, and social efficiency (price of anarchy). The theory of contests has been developed over years in the context of the economic theory, including public choice and political economy, operations research, and more recently, computer science. Competition-based incentives have been used over centuries to solicit innovations and they constitute a significant part of the design of modern online services, e.g. the use of crowdsourcing platforms and referral incentives.

The course will cover some of the following topics:

(a) Standard all-pay contest: complete information game, non-existence of pure-strategy Nash equilibrium, existence and full characterization of mixed-strategy Nash equilibria, exclusion principle, caps; incomplete information game, existence and uniqueness of a symmetric Bayes-Nash equilibrium, revenue equivalence.

(b) Rank order allocation of prizes: complete information game and multiple prizes, multiple identical prizes; incomplete information game and conditions for optimality of rewarding a single prize, sensitivity on the shape of production cost functions, status prizes.

(c) Smooth allocation of prizes: contest success functions, axiomatic and uncertainty justification of particular families of contest success functions, general logit contest success functions, existence and uniqueness of pure-strategy Nash equilibrium under general logit contest success functions, existence and characterization of pure-strategy Nash equilibrium under ratio-form, proportional, weighted proportional, and weighted valuation prize allocations, optimal contest success functions, difference-form contest success functions.

(d) Simultaneous contests: complete information game of simultaneous standard all-pay contests, the Colonel Blotto game, incomplete information game of simultaneous standard all-pay contests, existence, uniqueness, full characterization of equilibrium, simultaneous contests with proportional prize allocation.

(e) Sequential contests: equilibrium properties of contests with sequential allocation of prizes, comparison with single grand contest with simultaneous moves, multiple-round contests, sequential allocation of a prize with a termination rule.

(f) Public goods: tragedy of commons, complete information game of utility sharing, egalitarian sharing, proportional sharing, smoothness framework for establishing price of anarchy bounds, the effect of production cost functions.

(g) Tournaments: seeding of tournaments, standard seeding procedure, random permutation seeding, randomized cohort seeding, dynamic seeding procedures, desirable properties of delayed confrontation, monotonicity and envy freeness, strategic theory of tournaments, optimal prize split across stages.

(h) Referral prizes: allocation rules, strong and weak referral incentives, impossibility results, Sybil attacks, fixed payment contracts, split contracts, Nash equilibrium characterization, sufficient prize purse for given reachability.

(i) Rating systems: probabilistic rating model, the model of paired comparisons, maximum likelihood inference, existence and uniqueness of maximum likelihood estimates, Bayesian inference, factor graphs, approximate assumed density filtering, Gibbs sampling, rating systems Elo, Glicko, TopCoder, and TrueSkill.

(j) Information labeling: simple majority decoding, weighted majority decoding, optimality of weighted majority rule, optimal assignment of labeling tasks to workers.

Desirable previous knowledge

Familiarity with the basic concepts of game theory will be useful but not assumed, e.g. those covered in the course Mathematics of Operational Research MMath/MASt (Part III).

Introductory reading

(a) V. Krishna, Auction Theory, Academic Press (2002).

(b) E. Law and L. von Ahn, Human Computation, Morgan & Claypool (2011).

(c) N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, Algorithmic Game Theory, Cambridge University Press (2007).

(d) M. Osborne and A. Rubenstein, A Course in Game Theory, MIT Press (1994).

Particle Physics, Quantum Fields and Strings

The courses on Symmetries, Fields and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of the course Symmetries, Fields and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year		*Courses*
Second	*Essential*:	Quantum Mechanics, Methods, Complex Methods.
	Helpful:	Electromagnetism.
Third	*Essential*:	Principles of Quantum Mechanics, Classical Dynamics.
	Very helpful:	Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Field Theory (M24)

Professor M. J. Perry

Quantum Field Theory is the language in which all of modern physics is formulated. It represents the unification of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, Dirac spin-1/2), and massless gauge fields representing the photon. The relativistic invariance and symmetry properties of these fields iwll be discussed using the Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is developed in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained. Path integrals will then be introduced as an alternative way of seeing how free fields can be described.

Interactions are introduced using perturbative techniques and the role of Feynman diagrams is explained. This is first illustrated for theories with a purely scalar field interaction, and then for a Yukawa coupling between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processed are computed.

If there is time at the end of the course, we will briefly discuss some of the difficulties of including loops into Feynman diagrams.

Necessary Previous Knowledge

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Books

(a) M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press, (1995).

(b) S. Weinberg, *The Quantum Theory of Fields Vol I*, Cambridge University Press (1995)

(c) A. Zee, *Quantum Field Theory in a Nutshell*, (second edition), Princeton University Press, (2010)

(d) M. Srednicki, *Quantum Field Theory*, Cambridge University Press, (2007).

Symmetries, Fields and Particles (M24)

N. S. Manton

The course starts with a brief introduction to the various types of elementary particle – quarks, leptons, gauge and Higgs particles – and to the various symmetry groups that are useful for classifying the particles and understanding their properties. Some symmetry groups in particle physics are exactly realised, and some only approximately. The most important groups are certain specific Lie groups, including SU(2), SU(3) and the Lorentz and Poincaré groups.

A presentation of the basics of Lie group theory is given, including a discussion of Lie algebras and the relation of a Lie group to its Lie algebra. Representations of Lie groups and Lie algebras are defined, and the relationship between them also discussed. The representation theory of SU(2), which is closely related to quantum mechanical angular momentum theory, is extended to the case of SU(3) representations. Hadrons (particles containing quarks and antiquarks) are classified by representations of SU(3) based on an approximate flavour symmetry among quarks.

The detailed theory of elementary particles – the Standard Model – is a gauge theory, that is, a quantum field theory with an exact, locally acting Lie group symmetry. An introduction to the structure of the Lagrangian of a gauge theory will be given, including an introduction to how the gauge symmetry may be spontaneously broken by the Higgs mechanism. This is developed in more detail in the Standard Model course.

The course ends with a discussion of the Lorentz and Poincaré groups, and their representations, and how these are used to classify the momentum and spin states of relativistic particles.

The course will be backed up by examples sheets and examples classes. It is designed to be taken in conjunction with the Quantum Field Theory course, although either can be taken independently.

Desirable Previous Knowledge

Basic theory of finite groups, subgroups, orbits. Knowledge of some matrix Lie groups is also useful. Lie groups are differential manifolds, so some basic knowledge of manifold theory (coordinates, dimension, tangent spaces) is helpful. Special relativity and basic quantum theory will be assumed known, including orbital angular momementum theory, Pauli spin matrices, and tensor product rules for combining angular momenta.

Introductory Reading

(a) Perkins, D.H., Introduction to High Energy Physics, 4th ed., CUP (2000).

Reading to complement course material

(a) Costa, G. and Fogli, G., Symmetries and Group Theory in Particle Physics, Lecture Notes in Physics 823, Springer (2012).

(b) Jones, H.F., Representations and Physics, 2nd ed., Taylor and Francis (1998).

(c) Georgi, H., Lie Algebras in Particle Physics, Westview Press (1999).

(d) Fuchs, J. and Schweigert, C., Symmetries, Lie Algebras and Representations, paperback ed., CUP (2003).

Statistical Field Theory (M16)

R R Horgan

This course is an introduction to the renormalization group, the basis for a modern understanding of field theory, and the construction of effective field theories. The discussion is concerned with statistical systems including their relationship with quantum field theory in its Euclidean formulation.

The phenomenology of phase transitions is reviewed, leading to the introduction of the theory of critical phenomena. Landau-Ginsburg theory and mean field theory are presented and applied to the Ising model. The classification of phase transitions and their relationship with critical points is presented, and the renormalization group is introduced first in the context of the soluble 1D Ising model and then in general. The renormalization group is used for calculating properties of systems near a phase transition, for example in the Ising and Gaussian models, and the concepts of critical exponents, anomalous dimensions, and scaling are discussed.

The idea of the continuum limit for models controlled by a critical point and the relationship with continuum quantum field theory is elucidated.

Perturbation theory is introduced for the scalar field model with interactions and some example calculations are presented.

A background knowledge of Statistical Mechanics at an undergraduate level is essential. Although not a formal prerequisite, attendance at the Part III course Quantum Field Theory is a considerable advantage and strongly advised.

Books

(a) J M Yeomans, *Statistical Mechanics of Phase Transitions*, Oxford Scientific Publishing (1992)

(b) L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press (1996)

(c) J Cardy, *Scaling and Renormalization in Statistical Physics*, Cambridge Lecture Notes in Physics (1996)

(d) J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992)

(e) D Amit and V Martín-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005)

(f) C Itzykson and J-M Drouffe, *Statistical Field Theory*, Vols. 1-2, Cambridge University Press (1991)

Supersymmetry (L16)

B.C. Allanach

This course provides an introduction to the use of supersymmetry in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables and relates fermions and bosons.

Firstly, a physical motivation for supersymmetry is provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced.

Three examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or be familiar with the material covered in them.

Introductory Reading

(a) The first chapters of `http://arxiv.org/abs/hep-ph/0505105`

Reading to complement course material

For more advanced topics later in the course, it will helpful to have a knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model in order to understand the final lecture on the minimal supersymmetric standard model.

Beware: most of the supersymmetry references contain errors in minus signs, aside (as far as I know) Wess and Bagger.

(a) Course lecture notes from last year: `http://www.damtp.cam.ac.uk/user/examples/3P7.pdf`

(b) Videos of a very similar lecture course: follow the links from `http://users.hepforge.org/~allanach/teaching.html`

(c) Supersymmetric Gauge Field Theory and String Theory, Bailin and Love, IoP Publishing (1994) has nice explanations of the physics. An erratum can be found at `http://www.phys.susx.ac.uk/~mpfg9/susyerta.htm`

(d) Introduction to supersymmetry, J.D. Lykken, `hep-th/9612114`. This introduction is good for extended supersymmetry and more formal aspects.

(e) Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this terse and more mathematical book has the opposite sign of metric to the course.

(f) A supersymmetry primer, S.P. Martin, `hep-ph/9709256` is good and detailed for phenomenological aspects, although with the opposite sign metric to the course.

Advanced Quantum Field Theory (L24)

DB Skinner

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalisation of electrodynamics and form the backbone of the Standard Model - our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantising a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson Loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent. A further major component of the course is to study Renormalization. Wilsons picture of Renormalisation is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the Renormalisation Group (RG) flow. The course explores renormalisation systematically, from the use of dimensional regularisation in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as asymptotic freedom, this phenomenon revolutionised our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrise possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Time permitting, we may also discuss various modern topics in QFT, such as dualities, localization and topological QFTs,

Desirable Previous Knowledge

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

Introductory Reading

(a) Zee, A., *Quantum Field Theory in a Nutshell*, 2$^{\text{nd}}$ edition, PUP (2010).

Recommended Books

(a) Srednicki, M., *Quantum Field Theory*, CUP (2007).

(b) Weinberg, S., *The Quantum Theory of Fields*, vols. 1 and 2, CUP (1996).

(c) Banks, T. *Modern Quantum Field Theory: A Concise Introduction*, CUP (2008).

(d) Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).

Standard Model (L24)

M.B. Wingate

The Standard Model of particle physics is, by far, the most successful application of quantum field theory. As this booklet goes to press, this model accurately describes all experimental measurements involving strong, weak, and electromagnetic interactions.

The Standard Model is the quantum theory of the gauge group $SU(3) \times SU(2) \times U(1)$ with fermion fields for the leptons and quarks. The course aims to demonstrate how this model is realised in nature. It is intended to complement the more general Advanced QFT course.

This course begins by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content in terms of spin 1/2 leptons and quarks and also the spin 1 gauge bosons. The parity P, charge conjugation C and time-reversal T transformation properties of the theory are investigated. These need not be symmetries manifest in nature; e.g. only left-handed particles feel the weak force in violation of parity symmetry. We show how CP violation becomes possible when there are three generations of particles.

Ideas of spontaneous symmetry breaking are applied to discuss the Higgs Mechanism; the weakness of the weak force is due to the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry. The recent measurements of what appear to be Higgs boson decays will be presented.

We show how to obtain cross-sections and decay rates from the matrix element squared of a process. Various scattering and decay processes can be calculated in the electroweak sector using perturbation theory because of the smallness of the couplings. We touch upon the topic of neutrino masses and oscillations, an important window into physics beyond the Standard Model.

The strong interactions are based upon the gauge theory with (unbroken) gauge group $SU(3)$, called quantum chromodynamics (QCD). At low energies quarks are confined, forming bound states called hadrons. In such a non-abelian theory, the coupling constant decreases in higher energy processes to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. Nonperturbatively, progress can be made in the limits of very small and very large quark masses, making use of chiral and heavy quark symmetries. We introduce the framework of effective field theory and apply it to QCD.

Very high energy experiments and very precise experiments are currently striving to observe effects not describable by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model.

Four examples sheets and classes complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or to be familiar with the material covered in them. It is advantageous to attend the Advanced Quantum Field Theory course during the same term as attending this course, or to study renormalisation and non-abelian gauge fixing.

Reading to complement course material

(a) M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley (1996).

(b) H. Georgi, Weak Interactions, Benjamin/Cummings (1984).

(c) T-P. Cheng and L-F. Li, Gauge Theory of Elementary Particle Physics, Oxford University Press (1984).

(d) I.J.R. Aitchison and A.J.G. Hey, Gauge Theories in Particle Physics, IoP Publishing (1989).

(e) F. Halzen and A.D. Martin, Quarks and Leptons: An Introductory Course in Modern Particle Physics, John Wiley and Sons (1984).

(f) J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model, Cambridge University Press (1994).

String Theory (L24)

P. Townsend

The basic idea of String Theory is that elementary particles are excitations of a relativistic string, which could be open (i.e. with two endpoints) or closed. Each quantum excitation of the string behaves like an elementary particle, and closed strings have a massless spin-2 particle in their spectrum, which suggests that String Theory is a theory of quantum gravity. Open strings yield analogous generalisations of gauge theory, so a theory of open and closed strings is potentially one that can unify gravity with the forces of the standard model of particle physics.

This course will introduce the strings of string theory as constrained dynamical systems, focusing on the Nambu-Goto string. Various methods of quantisation of the free string will be discussed, including light-cone gauge, "old covariant" and BRST quantisation; this will reveal that there is a critical space-time dimension (26 for the Nambu-Goto string), and that the ground state is a tachyon. A study of the possible boundary conditions on open strings will reveal that any complete String Theory should include branes as well as strings. Some aspects of superstring theories will also be discussed: why there is no tachyon, why the critical dimension is 10, and why there are five of them.

An introduction to basic ideas of conformal field theory, and their application to String Theory will be given, along with a discussion of some aspects of interacting strings: computations of the simplest amplitudes (Veneziano and/or Virasoro-Shapiro formulas) and general principles of perturbation theory that follow from a path-integral approach. This will lead to the idea of an effective action for the massless particles, either in space-time (where we get generalisations of General Relativity) or on a brane (where we get a generalisation of gauge theory). This will provide the basis for a brief discussion (non-examinable) of how the five superstring theories are unified by an 11-dimensional "M-Theory".

Introductory Reading

(a) Green, Schwarz and Witten, "Superstring Theory: Vol. 1:Introduction" (CUP 1987)

(b) Brink and Henneaux, "Principles Of String Theory" (Plenum 1988).

(c) Lust and Theisen, "Lecture Notes in Physics: Superstring Theory" (Springer 1989)

Classical and Quantum Solitons. (E 16)

N. Dorey

Solitons are solutions of the classical field equations with particle-like properties. In particular, they are localised in space, have finite energy and are stable against decay into radiation. After quantisation, they give rise to new particle states which are typically very massive at weak coupling but can become light at strong coupling. Solitons play a key role in many recent advances in field theory and string theory, especially in the phenomenon of duality which relates the strong-coupling behaviour of one theory to the weak-coupling behaviour of another. In this course we will study the properties of classical solitons and their quantum counterparts. We will focus mainly on the case of integrable theories in two dimensional spacetime where an exact analytic description is possible.

Desirable Previous Knowledge

Quantum Field Theory. Advanced Quantum Field Theory.

Introductory Reading

(a) Topological Solitons, N. Manton and P. Sutcliffe (CUP 2004), Chapters 1, 4 and 5

Advanced String Theory (E16)

Professor M. J. Perry

This course follows on directly from Prof. Townsend's Lent Term course: String Theory. It is antcipated that will look in detail at the following topics: strings in curved spacetimes, branes, duality and compactification. The topics in the exam will be defined entirely by the material lectured.

Necessary Previous Knowledge

You will need to be fairly familiar with the contents of the following Part III courses or their equivalents; quantum field theory, advanced quantum field theory, string theory and a first course in general relativity.

Books

(a) R. Blumenhagen, D. Lüst and S. Theisen, "Basic Concepts in String Theory," Springer, 2013.

Relativity and Gravitation

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year		*Courses*
First	*Essential*:	Vectors & Matrices, Diff. Eq., Vector Calculus, Dynamics & Relativity.
Second	*Essential*:	Methods, Quantum Mechanics, Variational Principles.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Very helpful*:	General Relativity, Stat. Phys., Electrodynamics, Classical Dynamics, Cosmology.
	Helpful:	Further Complex Methods, Asymptotic methods.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Cosmology (M24)

Daniel Baumann

Cosmology has become a precision science. The basic Big Bang picture provides quantitative explanations for the expansion of the universe, the origin of the cosmic microwave background radiation, the synthesis of light chemical elements and the formation of stars, galaxies and large-scale structures. Moreover, there is growing evidence that all of the large-scale structures we see around us originated from microscopic quantum fluctuations, stretched to cosmic sizes during a period of inflationary expansion. However, there are still important gaps in our understanding, including the nature of the dark matter, the cause of the observed late-time acceleration of the universe, the classic puzzle of the initial singularity and the physical origin of inflation.

This course will develop the standard Big Bang cosmology and review its major successes and some of the challenges now faced at the cutting-edge of the field. We will emphasize the point of view that cosmology provides some of the best tests of modern ideas in particle physics.

Course website: www.damtp.cam.ac.uk/user/db275/Cosmology/

Desirable Previous Knowledge

Basic knowledge of relativity, quantum mechanics and statistical mechanics will be helpful. However, the course will be presented in a self-contained way, so students with less experience in any of these fields should have no problem to catch up.

Introductory Reading

(a) Weinberg, *The First Three Minutes.*
(b) Carroll, *A No-Nonsense Introduction to General Relativity.*

Reading to complement course material

(a) Dodelson, *Modern Cosmology.*
(b) Kolb and Turner, *The Early Universe.*
(c) Weinberg, *Cosmology.*
(d) Mukhanov, *Physical Foundations of Cosmology.*
(e) Peter and Uzan, *Primordial Cosmology.*

General Relativity (M24)

Ulrich Sperhake

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.
Course website: `http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html`

Desirable Previous Knowledge

This course will be self-contained, so previous knowledge of General Relativity is not essential. However, many students have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before then it is strongly recommended that you study an introductory book (e.g. Hartle or Schutz) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Familiarity with Newtonian gravity and special relativity is essential. Knowledge of the relativistic formulation of electrodynamics is desirable. Familiarity with finite-dimensional vector spaces, the calculus of functions $f : R^m \to R^n$, and the Euler-Lagrange equations will be assumed.

Introductory Reading

(a) Gravity: An introduction to Einstein's General Relativity, J.B. Hartle, Addison-Wesley, 2003.
(b) A First Course in General Relativity, B. Schutz, Cambridge University Press, 2009.

Reading to complement course material

(a) General Relativity, R.M. Wald, University of Chicago Press, 1984.
(b) Spacetime and Geometry: An Introduction to General Relativity, S.M. Carroll, Addison-Wesley, 2004.
(c) Advanced General Relativity, J.M. Stewart, Cambridge University Press, 1993.
(d) Introduction to General Relativity, L. Ryder, Cambridge University Press, 2009.
(e) 3+1 Formalism and Bases of Numerical Relativity, E. Gourgoulhon, 2007: `http://arxiv.org/abs/gr-qc/0703035`.

Chapter 1 of John Stewart's book gives a concise overview of differential geometry which also guides this part of the course. Carroll's and Ryder's books are very readable introductions. Gourgoulhon's notes provide a comprehensive overview of the space-time split of general relativity. Wald's book discusses many advanced topics; very suitable for obtaining comprehensive treatment on isolated topics.

Numerical General Relativity (M16)

Pau Figueras and Helvi Witek

General Relativity (GR) is our most succesful theory of gravity to date and finds applications in astrophysics and cosmology as well as high energy physics. Understanding gravity boils down to studying Einstein's equations, a set of non-linear, coupled partial differential equations (PDEs). There are many interesting problems including finding new types of stationary black hole solutions and investigating highly dynamical systems, such as the 2-body problem in GR, which cannot be solved analytically and instead require a numerical treatment.

The first part of this course will be devoted to a general introduction to the PDEs that more often arise in mathematical physics as well as the numerical methods to solve them. We will introduce finite difference methods, spectral methods and (time permitting) finite element methods.

In the second part of the course we will discuss how to solve the Einstein equations for stationary (i.e., time independent) problems. We will first discuss how one can formulate the problem in a way such that the equations are manifestly elliptic and can, therefore, be solved as a standard boundary value problem. We will introduce the general techniques to solve such problems, namely Ricci flow and Newton's method, through practical examples.

The third part of this course will focus on the time evolution problem in GR, which allows to explore highly dynamical systems in the strong curvature regime. We will derive the formulation of Einstein's equations as Cauchy problem and describe techniques to solve for the initial data and its evolution in time. We will discuss the numerical stability of evolution schemes which is closely connected to the well-posedness of the underlying PDE system. We will further present the BSSN and Generalized Harmonic formulations of the initial value problem which are widely used throughout the Numerical Relativity community. If time permits, we will discuss extensions of the "standard" Numerical Relativity techniques in 4-dimensional, asymptotically flat spacetimes to higher dimensions and more generic asymptotics.

Pre-requisite Mathematics

Pre-requisites include knowledge of General Relativity (at undergraduate level) and the theory of partial differential equations. Programming skills and familiarity with computer algebra programs such as mathematica are not mandatory but would be useful.

Literature

The following texts will cover the majority of the course, and are available online.

A. Iserles, *A first course in the numerical analysis of differential equations*, Cambridge Texts in Applied Mathematics, CUP, 2008.

Ll. Trefethen, *Spectral Methods in Matlab*, SIAM, 2000.

T. Wiseman, *N*umerical construction of static and stationary black holes, arXiv:1107.5513 [gr-qc].

M. Alcubierre, *Introduction to 3+1 numerical relativity* , International series of monographs on physics, Oxford Univ. Press, Oxford, 2008;

T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity*, Cambridge University Press, 2010;
see also `http://arxiv.org/abs/gr-qc/0211028`

E. Gourgoulhon *3+1 formalism and bases of numerical relativity*, 2007,

`http://arxiv.org/abs/gr-qc/0703035`

H. Witek, *Lecture Notes: Numerical Relativity in higher dimensional spacetimes*, International Journal of Modern Physics A Vol. **28**, 1340017 (2013);
see also `http://arxiv.org/abs/arXiv:1308.1686`

D. Hilditch, *An Introduction to Well-posedness and Free-evolution*, International Journal of Modern Physics A Vol. **28**, 1340015 (2013);
see also `http://arxiv.org/abs/arXiv:1309.2012`

Applications of Differential Geometry to Physics. (L16)

Maciej Dunajski

This is a course designed to develop the Differential Geometry required to follow modern developments in Theoretical Physics. The following topics will be discussed.

- Differential Forms and Vector Fields.
  - (a) One parameter groups of transformations.
  - (b) Vector fields and Lie brackets.
  - (c) Exterior algebra.
  - (d) Hodge Duality.
- Geometry of Lie Groups.
  - (a) Group actions on manifolds.
  - (b) Homogeneous spaces and Kaluza Klein theories.
  - (c) Metrics on Lie Groups.
- Fibre bundles and instantons.
  - (a) Principal bundles and vector bundles.
  - (b) Connection and Curvature.
  - (c) Twistor space.

Desirable Previous Knowledge

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

Reading to complement course material

(a) `http://www.damtp.cam.ac.uk/research/gr/members/gibbons/gwgPartIII_DGeometry2011-1.pdf`

(b) Flanders, H. Differential Forms. Dover

(c) Dubrovin, B., Novikov, S. and Fomenko, A. Modern Geometry. Springer

(d) Eguchi, T., Gilkey, P. and Hanson. A. J. Physics Reports 66 (1980) 213-393

(e) Arnold. V. Mathematical Methods of Classical Mechanics. Springer.

(f) Dunajski. M. Solitons, Instantons and Twistors. OUP.

Black Holes (L24)

H.S. Reall

A black hole is a region of space-time that is causally disconnected from the rest of the Universe. The study of black holes reveals many surprising and beautiful properties, and has profound consequences for quantum theory. The following topics will be discussed:

(a) Gravitational collapse. Why black holes necessarily form under certain circumstances.

(b) Causal structure, asymptotic flatness, Penrose diagrams, the event horizon.

(c) Explicit black hole solutions: Schwarzschild, Reissner-Nordstrom and Kerr solutions.

(d) Energy, angular momentum and charge in curved spacetime.

(e) The laws of black hole mechanics. The analogy with laws of thermodynamics.

(f) Quantum field theory in curved spacetime. The Hawking effect and its implications.

Examples sheets will be distributed during the course. Examples classes will be held to discuss these.

Desirable Previous Knowledge

Familiarity with the contents of the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Introductory Reading

(a) R.M. Wald, *General Relativity* (University of Chicago Press, 1984), Chapter 6.

Reading to complement course material

(a) P.K. Townsend, *Black holes: lecture notes*, arXiv:gr-qc/9707012.
(b) R.M. Wald, *General relativity*, University of Chicago Press, 1984.
(c) *Spacetime and Geometry*, S.M. Carroll, Addison Wesley, 2004.
(d) S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
(e) V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.
(f) N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
(g) R.M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Advanced Cosmology (L24)

E.P.S. Shellard and A. Challinor

This course will take forward at much greater depth the topics in modern cosmology covered at the end of the Michaelmas Term course. The prediction from fundamental theory for the primordial perturbation spectrum remains the key area of confrontation with cosmological observations, both from large-scale structure and the cosmic microwave sky. This course will develop the mathematical tools and physical understanding necessary for research in this very active area. If time permits we will also consider applications for specific models of particular current interest*.

Cosmological Perturbation Theory

- The 3+1 formalism and the Einstein equations
- Linearised Einstein equations for an expanding universe
- Review of density perturbation theory, transfer functions etc

Cosmic Microwave Sky

- Relativistic kinetic theory
- Collisionless Boltzmann equation
- Photon scattering and diffusion
- The CMB temperature power spectrum
- CMB Polarization

Topical issues: Non-Gaussianity from Inflation, Gravitational Waves

- "In-in" formalism and higher order correlation functions
- Non-Gaussianities from inflation
- Prospects for observations of non-Gaussianity
- Cosmological sources of stochastic gravitational waves
- Detection of cosmological gravitational waves
- Signatures of brane inflation and extra dimensions*

Pre-requisites

Familiarity with introductory Quantum Field Theory is recommended.

Course texts

Mukhanov, V. *Physical Foundation of Cosmology*, Cambridge (2005).

Bardeen, J.M., *Cosmological Perturbations From Quantum Fluctuations To Large Scale Structure*, DOE/ER/40423-01-C8 Lectures given at 2nd Guo Shou-jing Summer School on Particle Physics and Cosmology, Nanjing, China, Jul 1988. (Available on request).

Chen, X., *Primordial Non-Gaussianities from Inflation Models*, arxiv:1002.1416.

Dodelson, S., *Modern Cosmology*, Elsevier 2003.

Weinberg, S. *Cosmology*, Oxford University Press 2008.

Useful references

Efstathiou, G., in Physics of the Early Universe, Proc. 36th Scottish Summer School, eds. J. Peacock, et al., p. 361, Adam Hilger (1990).

Kolb, E.W. & M.S. Turner, *The Early Universe*, Addison-Wesley (1990).

Liddle, A. & Lyth, D., *Cosmological Inflation and Large Scale Structure*, Cambridge (2000).

Linde, A., *Particle Physics and Inflationary Cosmology*, Harwood (1990).

Ma, C., & Bertschinger, E., *Cosmological Perturbation Theory in Synchronous and Conformal Newtonian Gagues*, *Astrophysical Journal* **455**, 7 (1995) [astro-ph/9506072].

Misner, C.W., Thorne, K.S., and Wheeler, J.A., *Gravitation*, Freeman, 1973.

Mukhanov, V., Feldman, H., & Brandenberger, R., Phys. Reports 205, 203 (1992).

Padmanabhan, T., *Structure Formation in the Universe*, Cambridge (1993).

Peacock, J.A., *Cosmological Physics*, Cambridge (1999).

Peebles, P.J.E.,*Principles of Physical Cosmology*, Princeton (1993).

Weinberg, S.,*Gravitation and Cosmology*, Wiley (1973).

Spinor Techniques in General Relativity: Part 1 (L12)

Non-Examinable (Graduate Level)

Irena Borzym

The 2-spinor formalism has many applications and often provides a useful method of simplifying calculations which are quite cumbersome in terms of spacetime tensor language. The presentation will not assume previous familiarity with spinors. This is a graduate course so there will not be any suporting supervisions, instead the course will include many worked examples and illustrations.

An outline of the course is as follows.

(a) Two component spinors, their algebra and interpretation.

(b) Conformal group on Minkowski space.

(c) Translating between spinor and the more usual spacetime tensorial formulations.

(d) Simple geometric applications of spinors.

(e) Zero rest mass field equations.

(f) Conformal rescaling and transformation formulae.

(g) Petrov Classification with application to Maxwell tensor and Riemann tensor.

(h) Conformal compactification of Minkowski space.
(i) Geometry of Scri.
(j) Comparison with other constructions of Scri and compactification.
(k) Plucker Embedding.
(l) Comparison with Euclidean spacetime.

Desirable Previous Knowledge

The Part 3 general relativity course is a prerequisite.

Introductory Reading

(a) L. P. Hughston and K. P. Tod Freeman, Introduction to General Relativity. 1990.
(b) C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation.

Reading to complement course material

(a) J.M. Stewart, Advanced General Relativity. CUP, 1993.
(b) S. Ward and Raymond O. Wells Twistor Geometry and Field theory
(c) S. A Huggett and P. Tod. Introduction to Twistor Theory
(d) Penrose and Rindler Spinors and Spacetime Volume 1
(e) R.M. Wald, General Relativity. Chicago UP, 1984.
(f) S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Spacetime. CUP, 1973.
(g) Additional more specific references will be given in the lecture notes.

Spinor Techniques in General Relativity: Part 2. (E 12)

Non-Examinable (Graduate Level)

DR P.J. O'DONNELL

This course follows on from the Part 1 course given in Lent. The main emphasis will be placed upon further developing spinor techniques and applying these techniques to areas where tensor application becomes unwieldy or impossible.

The lectures will concentrate on the following topics:

Newman-Penrose (NP) spin coefficient formalism; NP field equations; NP quantities under Lorentz transformations; Geroch-Held-Penrose (GHP) formalism; Modified GHP formalism; Goldberg-Sachs theorem; Lanczos potential theory; Introduction to twistors.

Essential Previous Knowledge

Part III course in General Relativity. Part III course in Spinor Techniques in General Relativity: Part 1

Introductory Reading

(a) Rindler, W. (1966). What Are Spinors? Am. J. Phys. 34, pp937-942.

Reading to complement course material

(a) O'Donnell, P. (2003). Introduction to 2-spinors in general relativity. World Scientific.

(b) Penrose, R and Rindler, W. (1986). Spinors and Space-Time Vol. 2: Spinor and twistor methods in space-time geometry. Cambridge University Press.

(c) Chandrasekhar, S. (2000). The mathematical theory of black holes. Oxford University Press.

Astrophysics

Astrophysical Fluid Dynamics. (M24)

John Papaloizou

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. While many fluid dynamical effects can be seen in nature or the laboratory, there are other phenomena that are peculiar to astrophysics, for example self-gravitation, the dynamical influence of the magnetic field that is frozen in to a highly conducting plasma, and the dynamo effect driven by electromagnetic induction in a resistive fluid. The basic physical ideas introduced and applied in this course are those of Newtonian gas dynamics and magnetohydrodynamics (MHD) for a compressible fluid. The aim of the course is to provide familiarity with the basic phenomena and techniques that are of general relevance to astrophysics. Wherever possible the emphasis will be on simple examples, physical interpretation and application of the results in astrophysical contexts.

Examples of topics likely to be covered:

Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation. Microphysical basis and validity of a fluid description. Physical interpretation of MHD, with examples of basic phenomena, including dynamo theory. Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem. Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities. Spherically symmetric steady flows: stellar winds and accretion. Axisymmetric rotating magnetized flows: astrophysical jets. Waves and instabilities in stratified rotating astrophysical bodies.

Desirable Previous Knowledge

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of fluid dynamics, thermodynamics and electromagnetism will be assumed.

Introductory Reading

(a) Choudhuri, A. R. (1998). The Physics of Fluids and Plasmas. Cambridge University Press.

Reading to complement course material

(a) M.J.Thompson. An Introduction to Astrophysical Fluid Dynamics (2006). Imperial College Press.

(b) Landau, L. D., and Lifshitz, E. M. (1987). Fluid Mechanics, 2nd ed. Pergamon Press.

(c) Pringle, J. E., and King, A. R. (2007). Astrophysical Flows. Cambridge University Press.

(d) Shu, F. H. (1992). The Physics of Astrophysics, vol. 2: Gas Dynamics. University Science Books.

Structure and Evolution of Stars (M24)

A.N.Żytkow

Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a mathematical description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These basic equations have to be supplemented by a number of appropriately chosen equations describing the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed. Some familiarity with the principles of hydrodynamics, thermodynamics, quantum mechanics, atomic and nuclear physics will be assumed.

Approximate solutions of the equations will be shown; polytropic gas spheres, homology principles, the virial theorem will be presented. The evolution of a star will be discussed, starting from the main-sequence, following the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

The only way in which we may test stellar structure and evolution theory is through comparison of the theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

There will be four example sheets each of which will be discussed during an examples class.

Desirable Previous Knowledge

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics although a detailed knowledge of all of these is not expected.

Introductory Reading

(a) Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.

(b) Phillips, A. *The Physics of Stars*, Wiley, 1999.

Reading to complement course material

(a) Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.

(b) Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

(c) Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution*, Springer-Verlag, 1990.

The Origin and Evolution of Galaxies (M24)

Martin Haehnelt

Galaxies are a fundamental building block of our Universe. The course will give an account of the physics of the formation of galaxies and their central supermassive black holes in the context of the standard paradigm for the formation of structure in the Universe.

Specific topics to be covered include the following:

- Observed properties of galaxies
- Cosmological framework and basic physical processes
- The interplay of galaxies and the intergalactic medium from which they form
- Numerical Methods for modeling galaxy formation
- Collapse of dark matter haloes and the inflow/outflow of baryons
- The hierarchical build-up of galaxies
- The origin and evolution of the central supermassive black holes in galaxies
- Towards understanding the origin of the Hubble sequence of galaxies

Desirable Previous Knowledge

The course is aimed at astronomers/astrophysicists (including beginning graduate students). It should be also suitable for interested physicists and applied mathematicians. The course is self-contained, but students who have previously attended introductory courses in General Relativity and/or Cosmology will have an easier start.

Introductory Reading

(a) Ryden, B., *Introduction to Cosmology*, 2003, Addison-Wesley.

(b) Sparke, L., Gallagher, J.S., *Galaxies in the Universe*, 2nd ed., 2007, Cambridge University Press.

Reading to complement course material

(a) Mo, H., van den Bosch, F., White, S., *Galaxy Formation and Evolution*, 2010, Cambridge University Press.

(b) Schneider, P., *Extragalactic Astronomy and Cosmology: An Introduction*, 2006, Springer.

(c) Coles P., Lucchin F., *Cosmology - The Origin and Evolution of Cosmic Structure* (second edition), 2002, Wiley.

Astronomy of Strong Gravity: Gravitational Wave Generation and Detection (L12)

Non-Examinable (Graduate Level)

Jonathan R. Gair and Priscilla Canizares

Gravitational wave (GW) astronomy is an emergent area of research, which relies on the development of new theoretical, numerical and technological tools that will be needed in the forthcoming advanced GW detector era.

The detection of GWs will not only provide us with a new tool to learn about the nature of the Universe, but will offer a way to unveil objects and regions of space that otherwise would remain hidden and unknown. In particular, this includes the strong gravitational field regions close to black holes, which are usually surrounded by matter which blocks the emission of electromagnetic radiation.

This course will discuss GW generation, propagation and detection, and how GWs will provide information about the structure and nature of the spacetime of their generating sources. We will discuss how we can distinguish which theory of gravity describes the strong field regime. We will put a particular focus on binary systems with extreme mass ratio, since they provide the most precise probes of spacetime. The mathematical techniques that will be descfibd in this course will include relativistic perturbation theory on flat and curved backgrounds, including black hole perturbation theory.

Pre-requisite Mathematics The course will assume a knowledge of general relativity at an advanced undergraduate level, similar to that of the Part II general relativity course. The part III GR course will be helpful, but not essential.

Literature

The following texts will cover the majority of the course, and are available either on the University library or online.

[1] Charles W. Misner, Kip S. Thorne and John Archibald Wheeler, *Gravitation*, San Francisco : W. H. Freeman, c1973.

[2] Wald, Robert M., *General Relativity*, University of Chicago Press, 1984.

[3] Sasaki, Misao and Tagoshi, Hideyuki, *Analytic Black Hole Perturbation Approach to Gravitational Radiation.*
`http://www.emis.ams.org/journals/LRG/Articles/lrr-2003-6/`

Dynamics of Astrophysical Discs (L16)

Henrik Latter

A disc of matter in orbital motion around a massive central body is found in numerous situations in astrophysics. For example, Saturn's rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a (non-Newtonian) fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary star systems and in the centres of active galaxies, where they can reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of extrasolar planets.

Provisional synopsis:

Occurrence of discs in various astronomical systems, basic physical and observational properties.

Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.

Evolution of an accretion disc.

Vertical disc structure, order-of-magnitude estimates and timescales, thin-disc approximations, thermal and viscous stability.

Shearing sheet, symmetries, shearing waves.

Incompressible dynamics: hydrodynamic stability, vortices.

Compressible dynamics (2D): density waves, gravitational instability.

Density waves in cylindrical geometry, Lindblad and corotation resonances.

Satellite-disc interaction, tidal potential, resonant torques, impulse approximation.

Magnetorotational instability.

Desirable previous knowledge

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is needed for the magnetorotational instability, but self-contained notes on this topic will be available.

Introductory reading

Much information on the astrophysical background is contained in

(a) Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.

Some of the basic theory of accretion discs is described in

(a) Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.

Reading to complement course material

(There are no suitable textbooks.)

Galactic Astronomy and Dynamics (L24)

N.W. Evans

Astrophysics provides many examples of complex dynamical systems. This course covers the mathematical tools to describe Galaxies as well as reviewing their observational properties. The behaviour of these systems is controlled by Newton's laws of motion and Newton's law of gravity. Galaxies are dynamically very young, a typical star like the Sun having orbited only thirty or so times around the galaxy. The motions of stars in Galaxies are described using classical statistical mechanics, since the number of stars is so great. The study of large assemblies of stars interacting via long-range forces provides many unusual examples of cooperative phenomena, such as bars and spiral structure. The interplay between astrophysical dynamics and modern cosmology is also important – much of the evidence for dark matter is dynamical in origin

A detailed syllabus is as follows:

Observational overview. Stellar populations in galaxies, galaxy morphology and classification. Dust and gas in galaxies. Scaling Laws.

Theory of the gravitational potential. Poisson's equation. The gravity field of spherical. elliptical and disk galaxies. Regular and chaotic orbits, the epicyclic approximation, surfaces of section, action-angle coordinates, adiabatic invariance.

Collisionless stellar dynamics, the Boltzmann equation, the Jeans theorem, the Jeans equations, equilibrium models of spherical, elliptical and disk galaxies. Collisions, collisional dynamics, the Fokker-Planck equation. Globular cluster evolution, evaporation and ejection, the gravothermal catastrophe, hard and soft binaries.

Galactic stability, The Jeans length, theories of spiral structure, the role of resonances. The Milky Way Galaxy, the thin disk, thick disk and halo, substructure and tidal streams.

Desirable Previous Knowledge

The course is self-contained and suitable for astronomer and applied mathematicians. A knowledge of classical mechanics, methods of mathematical physics, and statistical physics would be helpful.

Introductory Reading

(a) Sparke L., Galalgher J., Galaxies in the Universe: An Introduction, Cambridge University Press

Reading to complement course material

(a) Binney, J., and Merrifield, M., Galactic Astronomy, Princeton University Press, 1999
(b) Binney J., Tremaine S., Galactic Dynamics, Princeton University Press
(c) Bertin G., Dynamics of Galaxies, CUP
(d) Heggie D., Hut P., The Million Body Problem, CUP
(e) Spitzer L., The Dynamical Evolution of the Globular Clusters, Princeton University Press

Planetary System Dynamics (L24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of extrasolar planets (more than 900 are now known) and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems. Specific topics to be covered will be drawn from the following:

(a) Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation

(b) Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits

(c) Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits

(d) Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments

(e) Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems

(f) Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap

(g) Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide

(h) Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift

(i) Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces

Desirable Previous Knowledge

This course is self-contained.

Reading to complement course material

(a) Murray C. D. and Dermott S. F., Solar System Dynamics, Cambridge University Press, 1999

(b) Armitage P. J., Astrophysics of Planet Formation, Cambridge University Press, 2010

(c) de Pater I. and Lissauer J. J., Planetary Sciences, Cambridge University Press, 2010

(d) Valtonen M. and Karttunen H., The Three-Body Problem, Cambridge University Press, 2006

Quantum Computation, Information and Foundations

Quantum Information Theory (M24)

Nilanjana Datta

Quantum Information Theory (QIT) is an exciting, young field which lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the mathematical theory of acquisition, storage, transmission and processing of information. QIT is the study of how these tasks can be accomplished, using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical analogues. These new features can be exploited, not only to improve the performance of certain information-processing tasks, but also to accomplish tasks which are impossible or intractable in the classical realm.

This is an introductory course on QIT, which should serve to pave the way for more advanced topics in this field. The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of QIT. Topics in this part of the course will include a brief discussion of data compression, transmission of data through noisy channels, Shannon's theorems, entropy and channel capacity.

The quantum part of the course will commence with a study of open systems and a discussion of how they necessitate a generalization of the basic postulates of quantum mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs, the Kraus Representation Theorem and the Choi-Jamilkowski isomorphism. Entanglement and some applications elucidating its usefulness as a resource in QIT will be discussed. This will be followed by a study of the von Neumann entropy, its properties and its interpretation as the data compression limit of a quantum information source. Schumacher's theorem on quantum data compression will be discussed in detail. The definitions of ensemble average fidelity and entanglement fidelity will be introduced in this context. Definitions and properties of the quantum conditional entropy, quantum mutual information, the quantum relative entropy and the coherent information will be discussed. Various examples of quantum channels will be given and the different capacities of a quantum channel will be discussed. The Holevo bound on the accessible information and the Holevo-Schumacher-Westmoreland (HSW) Theorem will also be covered.

Pre-requisite Mathematics

Knowledge of basic quantum mechanics will be assumed. However, an additional lecture can be arranged for students who do not have the necessary background in quantum mechanics. Elementary knowledge of Probability Theory, Vector Spaces, Linear Algebra and Group Theory will be useful.

Literature

The following book and lecture notes provide interesting and relevant reading material.

1. M.A.Nielsen and I.L.Chuang, Quantum Computation and Quantum Information; Cambridge University Press, 2000.

2. M. M. Wilde, *From Classical to Quantum Shannon Theory*, http://arxiv.org/abs/1106.1445

3. J.Preskill, Chapter 5 of his lecture notes: Lecture notes on Quantum Information Theory, http://www.theory.caltech.edu/ preskill/ph229/

Quantum Computation (L16)

Richard Jozsa

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. Notable examples include integer factorisation (and consequent efficient breaking of commonly used public key crypto systems) and database searching. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

The course will cover the following topics:

Notion of qubits, quantum logic gates, circuit model of quantum computation. Basic notions of quantum computational complexity, oracles, query complexity.

The quantum Fourier transform. Exposition of fundamental quantum algorithms including the Deutsch-Jozsa algorithm, Shor's factoring algorithm and Grover's searching algorithm.

A selection from the following further topics:
(i) Quantum teleportation and the measurement-based model of quantum computation;
(ii) Lower bounds on quantum query complexity;
(iii) Applications of phase estimation in quantum algorithms;
(iv) Quantum simulation;
(v) Introduction to quantum walks.

Desirable Previous Knowledge

It is desirable to have familiarity with the basic formalism of quantum mechanics especially in the simple context of finite dimensional state spaces (state vectors, composite systems, unitary matrices, Born rule for quantum measurements). Revision notes will be provided giving a summary of the necessary material including an exercise sheet covering notations and relevant calculational techniques of linear algebra. It would be desirable for you to look through this material at (or slightly before) the start of the course.
Any encounter with basic ideas of classical theoretical computer science (complexity theory) would be helpful but is not essential.

Reading to complement course material

(a) Nielsen, M. and Chuang, I., Quantum Computation and Quantum Information. CUP.

(b) John Preskill's lecture notes on quantum information theory, available at
`http://www.theory.caltech.edu/people/preskill/ph219/`

(c) Andrew Childs lecture notes on quantum algorithms available at
`http://www.math.uwaterloo.ca/ amchilds/teaching/w11/qic823.html`

Quantum Foundations (L16)

Adrian Kent

In recent decades, there has been a renaissance of interest in foundational issues in quantum theory, particularly in relation to quantum information science, cosmology and quantum gravity. This course provides an introduction to modern research on quantum foundations.

We begin with an introduction to the Feynman path integral. We use toy models of path integrals to give a first discussion of the problem of giving a unified description of classical and quantum physics.

We then discuss quantum entanglement and the relationship between quantum theory and special relativity. We review the definitions of pure and mixed states, entanglement, and reduced density matrices. We then discuss Bell's theorem, the Clauser-Horne-Shimony-Holt and Braunstein-Caves inequalities, quantum non-locality, experimental tests of quantum non-locality and the failure of local hidden variable theories, and the delicate "peaceful co-existence" between quantum theory and the no-signalling principle in special relativity.

Finally in this section, we review the recent Pusey-Barrett-Rudolph theorem and its implications for our understanding of quantum theory.

In the second part of the course we return to the relationship between classical and quantum physics and review some of the major modern lines of research on this fundamental problem. We consider the physics of decoherence, some simple models of decoherence, and estimates of decoherence rates. This brings us to a more recent class of attempts at alternatives to quantum theory, the so-called 'dynamical collapse models' proposed by Ghirardi-Rimini-Weber, Pearle and others; we describe these models and review some of their problems. Finally, we discuss many-worlds quantum theory and the problem of making sense of probability in many-worlds theory.

Examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

A good understanding of undergraduate level quantum theory is required. (The Cambridge 1B Quantum Mechanics course is a good starting point.)

Optional Introductory Reading

(a) Benjamin Schumacher and Michael Westmoreland, Quantum Processes Systems, and Information, Cambridge University Press, Chapters 1-8. This is a good starting text for those students who wish to review the core aspects of quantum theory in the context of quantum information.

(b) M. Pusey, J. Barrett and T. Rudolph, Nature Physics 8, 476 (2012), available at arxiv:1111.3328 .

Optional reading and viewing to complement course material

(a) Benjamin Schumacher's lectures on Quantum Theory, archived at http://pirsa.org/C10028/, Adrian Kent's lectures on Quantum Theory, video archived at http://pirsa.org/C11018, and Robert Spekkens' lectures on Quantum Foundations, video archived at http://pirsa.org/C09040. These resources should probably be used selectively and sparingly. Note in particular that none of these video courses covers all the material in the present course, or in the same style.

(b) J.Preskill, Chapters 2,3,4 of his Lecture notes on Quantum Information Theory, available at http://www.theory.caltech.edu/ preskill/ph229/#lecture

(c) John Bell, "Speakable and Unspeakable in Quantum Mechanics" Cambridge University Press, 2nd edition, Chapters 1,2 and 22.

(d) Yakir Aharonov and Daniel Rohrlich, Quantum Paradoxes: Quantum Theory for the Perplexed, WILEY-VCH Verlag, Chapters 3, 7 and 14.

(e) "Many Worlds? Everett, Quantum Theory, and Reality", Simon Saunders, Jonathan Barrett, Adrian Kent and David Wallace (eds.) (Oxford University Press, 2010), in particular Chapter 8 and Chapter 10 (available at arXiv:0905.0624).

Advanced Quantum Information Theory. (L16)

Toby Cubitt

Quantum information theory is neither wholly physics (though it's almost entirely about quantum mechanics), nor wholly mathematics (though it's mainly concerned with proving rigorous mathematical results), nor wholly computer science (though much of it's to do with storing, processing, or transmitting information). Over the last two to three decades, it has developed into a rich mathematical theory of information in quantum mechanical systems, that draws on all three of these fields. More recently, this has been turned on its head: quantum information is beginning to be *used* to solve problems in physics, computer science, and mathematics.

The aim of this course is to select one or two advanced topics in quantum information theory, close to the cutting edge of research, and cover them in some depth and rigour.

This year, I will focus on quantum information in many-body systems. Quantum computation aims to engineer complex many-body systems to process information in ways that would not be possible classically. Many-body physics aims to understand the complex behaviour of naturally-occurring many-body systems. In a sense, they are opposite sides of the same coin. Recently, quantum information theory has been used both to prove important results in many-body physics, and to construct many-body models that exhibit very unusual physics, providing counterexamples to some of the standard intuition in condensed matter physics.

A possible outline (from which we may diverge to explore other related results) is as follows. We will begin by studying the computational complexity of quantum many-body systems, introducing the necessary complexity theoretic concepts along the way. An important milestone is Kitaev's proof of QMA-hardness of the ground state problem for local Hamiltonians. Kitaev's construction leads to systems with highly entangled ground states and polynomially-decaying spectral gap. So, in the second half of the course, we will turn to a quantum-information-inspired exploration of spectral gaps, correlations, and entanglement in many-body systems. Lieb-Robinson bounds, which limit the speed at which information can propagate in many-body systems, turn out to be a surprisingly useful tool for proving results about their static properties. We will see how they can be used to prove results relating spectral gaps, decay of correlations, and entanglement area laws (going into more or less detail, as time allows).

There will be examples sheets and examples classes to back up the lectures.

Desirable Previous Knowledge

Attendance of (or familiarity with the material from) the Michaelmas term "Quantum Information Theory" course is essential. A solid understanding of basic quantum mechanics will also be assumed. Attending the Lent term "Quantum Computation" course in parallel may be helpful, but is not necessary.

Introductory Reading

The reading material and lecture notes from the "Quantum Information Theory" course are also relevant to this course. The following cover the necessary background (and more):

(a) Schumacher, B. and Westmoreland, M. "Quantum Processes, Systems, and Information", Cambridge University Press.

(b) Nielsen, M. and Chuang, I., "Quantum Computation and Quantum Information", Cambridge University Press.

(c) John Preskill's lecture notes on quantum information theory, which are available at http://www.theory.caltech.edu/people/preskill/ph219/

Reading to complement course material

Most of the course material is not covered in any text book. The following may be helpful for some sections of the course:

(a) Kitaev, A., Shen, A., and Vyalyi M. "Classical and Quantum Computation", American Mathematical Society

(b) Aharonov, D. and Naveh, T. "Quantum NP – a Survey", http://arxiv.org/abs/quant-ph/0210077

(c) Matt Hastings' Les Houches summer school lecture notes, http://arxiv.org/abs/1008.5137

Philosophy of Physics

The courses in Philosophy of Physics are open to all students doing Part III, but are formally listed as graduate courses. This means there is no exam at the end of May for any such course; but a Part III student can get credit for them by doing their submitted Part III essay in association with one of the courses. More generally, the Philosophy of Physics courses are intended as a refreshing and reflective companion to the other Part III courses, especially the courses in theoretical physics.

Philosophical Foundations of Quantum Field Theory (M8)

Nazim Bouatta and Nicholas Teh

Quantum field theory (QFT) is a wonderful mountain range, combining strikingly deep and unifying ideas with a panoply of powerful calculational tools. In recent decades, QFT has become the framework for several basic and outstandingly successful physical theories. But it has been largely unexplored by philosophy of physics, which has concentrated on conceptual questions raised by non-relativistic quantum mechanics and general relativity (and the focus of another graduate course). Here, we will introduce the philosophical aspects of quantum field theory. More specifically, we will conceptually address topics that have been central to quantum field theory's development in the last forty years, such as: the renormalization group, gauge symmetries and solitons.

Desirable Previous Knowledge

There are no formal prerequisites. Previous familiarity with the tools of quantum field theory, such as provided by the Part III courses, will be helpful.

Introductory Reading

This list of introductory reading is approximately in order of increasing difficulty.

(a) Wallace, D. (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151** (1):33-80, 2006. Preprint available online at: http://arxiv.org/pdf/quant-ph/0112148v1

(b) Weinberg, S. (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: http://arxiv.org/abs/hep-th/9702027

(c) Fisher, M. (1998), 'Renormalization group theory: Its basis and formulation in statistical physics', *Rev. Mod. Phys* **70**, pp 653-681.

Reading to complement course material

(a) Cao, T. ed. The Conceptual Foundations of Quantum Field Theory. Cambridge University Press, 1999.

(b) Weinberg. S. The Quantum Theory of Fields, Vols I and II. Cambridge University Press, 1995 and 1996.

(c) Ruetsche, L. Interpreting Quantum Mechanics. Oxford University Press, 2011.

(d) Healey, R. Gauging What's Real. The Conceptual Foundations of Contemporary Gauge Theories. Oxford University Press, 2007.

Foundations of Dynamics and Relativity (L8)

J. Brian Pitts and David Sloan

Does Hamiltonian General Relativity really lack change? Must observable quantities in Hamiltonian GR be integrated over the whole universe? These surprising claims are often heard. The first half of the course will set up and evaluate these claims. But how does one get from a Lagrangian to a Hamiltonian if the Lagrangian doesn't permit a Legendre transformation, as electromagnetism, Yang-Mills, and GR do not? The standard answer, Dirac-Bergmann constrained Hamiltonian dynamics, will be discussed with attention to Maxwell, Einstein (often discarding spatial dependence), and at times Proca's massive electromagnetism. Emphasis will be placed on the equivalence of the Hamiltonian and Lagrangian formalisms, especially in relation to gauge transformations.

The second half of the course will consist of an application of the Hamiltonian formalism in the context of cosmology. This approach has proven useful within several contexts, including the analysis of inflation and the Mixmaster (Belinskii-Khalatnikov-Lifshitz) approach to generic space-like singularities. It also forms the basis of the canonical quantization used in Loop Quantum Gravity, a recent attempt to quantize Einstein's theory which has yielded an interesting cosmological sector. The Hamiltonian form of several classical cosmologies will be discussed within General Relativity, and a brief introduction given to Loop Quantum Cosmology.

Desirable Previous Knowledge

Knowledge of electromagnetism including its manifestly covariant form and derivation from a Lagrangian density will be assumed, as will a bit of Hamiltonian mechanics. Familiarity with General Relativity and cosmology at the Part II or Part III level would be helpful.

Introductory Reading

(a) J. Earman, *Philosophers' Imprint* **2** (2002), no. 3.

(b) T. Maudlin, *Philosophers' Imprint* **2** (2002) no. 4.

Reading to complement course material

(a) R. Wald, *General Relativity*, University of Chicago Press, appendix E.

(b) K. Sundermeyer, *Constrained Dynamics.* Excerpts.

(c) J. Anderson and P. Bergmann, *Physical Review* **83** (1951), p. 1018.

(d) L. Castellani, *Annals of Physics* **143** (1982), p. 357.

(e) J. Pons and D. Salisbury, *Physical Review D* **71** (2005), 124012, gr-qc/0503013.

(f) P. Bergmann, *Reviews of Modern Physics* **33** (1961), p. 510.

(g) T. Thiemann, *Modern Canonical Quantum General Relativity*, ch. 1.

(h) G. Gibbons and N. Turok, "The Measure Problem in Cosmology," hep-th/0609095.

(i) J. M. Heinzle, C. Uggla, and N. Rohr, "The Cosmological Billiard Attractor," gr-qc/0702141.

(j) I. Agullo and A. Corichi, "Loop Quantum Cosmology," to appear in *The Springer Handbook of Spacetime* (2013), arXiv:1302.3833.

Applied and Computational Analysis

Measure and Image (M16)

T. Valkonen

Photographs and other natural images are usually not smooth maps, they contain edges (discontinuities) and other non-smooth geometric features that should be preserved by image enhancement techniques. The correct mathematical modelling of these features involves the space of functions of bounded variation and, in consequence, aspects of geometric measure theory. The aim of this course is to provide an introduction to functions of bounded variation and their applications in image processing. It will cover the following topics.

- *Motivation.* Why Sobolev spaces are not enough for image processing? Functions of bounded variation of one variable.
- *Measure.* Refresher on measure theory. Hausdorff measure and rectifiable sets.
- *Functions of bounded variation.* Weak convergence and compactness. Poincaré inequality. Co-area formula. Fine properties.
- *Total variation regularisation.* Image denoising. Basic properties of solutions.
- *Special functions of bounded variation.* Compactness. The Mumford-Shah image segmentation problem.

This course is ideally complemented by the following courses:

- *Convex Optimisation with Applications to Image Processing* (M24) to learn about numerical methods for solving the problems studied analytically in this course.
- *Image Processing – Variational and PDE Methods* (L16) for further topics in image processing.

Desirable Previous Knowledge

A basic course in measure theory is strongly recommended, although we do include a quick refresher to the topic. Basic knowledge of Sobolev spaces and notions of weak convergence in function spaces are advantageous, but not necessary.

Introductory Reading

(a) A. Friedman, *Foundations of Modern Analysis*, Dover, 2003.

(b) W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987. (Part on measure theory)

(c) L. C. Evans, *Partial Differential Equations*, Americal Mathematical Society, 2010. (Part on Sobolev spaces)

Reading to complement course material

(a) L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000.

(b) G. Aubert, and P. Kornprobst, *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*, Springer, 2006.

(c) L. C. Evans, and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1991.

Convex Optimisation with Applications in Image Processing. (M16)

Jan Lellmann

Convex optimisation problems have the intriguing property that they can be numerically solved to a *global* minimum in a reasonable amount of time, even for many large-scale, non-linear, non-differentiable problems. With some effort many interesting real-world applications can be modeled by, or at least approximated by, the task of solving a single convex problem.

This approach has two strong points: firstly, if the method fails, we know that it is clearly not a numerical issue, but that in fact the model was wrong. Secondly, more often than not we find that the resulting algorithms are very efficient and scale almost linearly in the number of variables. This is especially important in image processing, where the number of variables can easily go into the millions for a single image.

Convex optimization as a field is now relatively mature, which makes for a very polished theory, but it still keeps evolving with the increasing computational power and new architectures such as GPUs. The number of applications in image processing is enormous – removing noise from digital camera images, increasing the resolution of an image, cutting out objects from the background, tracking people in video sequences, reconstructing 3D objects from several views, any many more.

The course is laid out as an introduction into the theory and solution strategies together with a collection of interesting applications in image processing and their specific challenges. We will begin with the theory in a conic optimization setting, including the fundamental results about subdifferentials, optimality conditions, and duality. We will then cover the most important solvers including Interior Point- and min-cut/max-flow methods, and recent first-order developments. Depending on time and interest we might also look into some complexity results.

Related courses

This course is ideally complemented by *Measure and Image (M16)* and *Image Processing – Variational and PDE Methods* (L16).

Desirable Previous Knowledge

A background in variational methods is helpful but not required, since we will mainly work in the finite-dimensional setting.

Introductory Reading

(a) S. Boyd, L. Vandenberghe: Convex Optimization. Cambridge University Press, 2004 (available online).

(b) R. T. Rockafellar, J.-B. Wets: Variational Analysis. Springer, 3rd ed., 2009.

(c) A. Ben-Tal, A. Nemirovski: Lectures on Modern Convex Optimization. MPS-SIAM, 2001.

(d) N. Paragios, Y. Chen, O. Faugeras: Handbook of Mathematical Models in Computer Vision. Springer, 2006.

Reading to complement course material

(a) Y. Nesterov: Introductory Lectures on Convex Optimization. Kluwer, 2004.

(b) D. P. Bertsekas: Network Optimization: Continuous and Discrete Models. Athena Scientific, 1998.

(c) J. Nocedal, S. J. Wright: Numerical Optimization. Springer, 2006.

(d) C. M. Bishop: Pattern Recognition and Machine Learning. Springer, 2006.

Numerical solution of differential equations (M24)

A. Iserles

The goal of this lecture course is to present and analyse efficient numerical methods for ordinary and partial differential equations. The exposition is based on few basic ideas from approximation theory, complex analysis, theory of differential equations and linear algebra, leading in a natural way to a wide range of numerical methods and computational strategies. The emphasis is on algorithms and their mathematical analysis, rather than on applications.

The course consists of three parts: methods for *ordinary differential equations* (with an emphasis on initial-value problems and a thorough treatment of stability issues and stiff equations), numerical schemes for *partial differential equations* (both boundary and initial-boundary value problems, featuring finite differences and finite elements) and, time allowing, *numerical algebra of sparse systems* (inclusive of fast Poisson solvers, sparse Gaussian elimination and iterative methods). We start from the very basics, analysing approximation of differential operators in a finite-dimensional framework, and proceed to the design of state-of-the-art numerical algorithms.

Desirable Previous Knowledge

Good preparation for this course assumes relatively little in numerical mathematics *per se,* except for basic understanding of elementary computational techniques in linear algebra and approximation theory. Prior knowledge of numerical methods for differential equations will neither be assumed nor is necessarily an advantage. Experience with programming and application of computational techniques will obviously aid comprehension but is neither assumed nor expected.

Fluency in linear algebra and decent understanding of mathematical analysis are a must. Thus, linear spaces (inner products, norms, basic theory of function spaces and differential operators), complex analysis (analytic functions, complex integrals, the Cauchy formula), Fourier series, basic facts about dynamical systems and, needless to say, elements from the theory of differential equations.

There are several undergraduate textbooks on numerical analysis. The following present material at a reasonable level of sophistication. Often they present material well in excess of the requirements for the course in computational differential equations, yet their contents (even the bits that have nothing to do with the course) will help you to acquire valuable background in numerical techniques:

Introductory Reading

(a) S. Conte & C. de Boor, *Elementary Numerical Analysis,* McGraw–Hill, New York, 1980.

(b) G.H. Golub & C.F. van Loan, *Matrix Computations,* 3rd edition. Johns Hopkins Press 1996.

(c) M.J.D. Powell, *Approximation Theory and Methods,* Cambridge University Press, Cambridge, 1981.

(d) G. Strang, *Introduction to Linear Algebra,* Wellesley-Cambridge Press, Cambridge (Mass.), 3rd ed. 2003..

Reading to complement course material

(a) U. M. Ascher, *Numerical Methods for Evolutionary Differential Equations,* SIAM, Philadelphia, 2008.

(b) O. Axelsson, *Iterative Solution Methods,* Cambridge University Press, Cambridge, 1996.

(c) E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer-Verlag, Berlin, 2nd ed. 1993.

(d) E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems,* Springer-Verlag, Berlin, 2nd ed. 1996.

(e) A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University Press, Cambridge, 2nd ed. 2008.

(f) K.W. Morton & D.F. Mayers, *Numerical Solution of Partial Differential Equations: An Introduction,* Cambridge University Press, Cambridge, 2005.

(g) G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Wellesley–Cambridge Press, Cambridge (Mass.), 2nd ed. 2008.

Distribution Theory and Applications (L16)

Dr A. Ashton

This course will provide an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will briefly look at Sobolev spaces in $\mathbf{R}^n$ and their description in terms of the Fourier transform of tempered distributions. Time permitting, the material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. We will also look at Hörmander's oscillatory integrals and use them to describe the singular support of a large class of distributions. The course will be supplemented with hand-outs and example sheets. There will be three examples classes.

Desirable Previous Knowledge

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Methods/Analysis). No knowledge of measure theory or functional analysis is required.

Introductory Reading

(a) F. G. Friedlander and M. S. Joshi, *Introduction to the Theory of Distributions*, Cambridge Univ Pr, 1998.

(b) M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge Univ Pr, 1958.

(c) G. B. Folland, *Introduction to Partial Differential Equations*, Princeton Univ Pr, 1995.

Reading to complement course material

(a) L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols I-II*, Springer Verlag, 1985.

(b) M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vols I-II*, Academic Press, 1979.

(c) F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Image Processing - Variational and PDE Methods (L16)

C.-B. Schönlieb

In our modern society the processing of digital images becomes more and more important, reflected in a myriad of applications: medical imaging (MRI, CT, PET), forensics, security, design, arts and many more. In this course we encounter some of the most powerful image processing methods and its underlying mathematical principles. In particular, we are interested in deterministic imaging approaches using variational calculus and partial differential equations (PDEs) [1-3].

The course is organised as follows: We start with mathematical representations of images (e.g., distributions, Sobolev functions, functions of bounded variation, level sets) and formulate inverse problems, i.e., optimization models, for image denoising, –decomposition, –inpainting and –segmentation (e.g., total variation minimization [4], Mumford-Shah, curve models, active contours). Then, we move on to PDEs for image processing (e.g., the heat equation, degenerate elliptic PDEs, Perona-Malik, diffusion filters, anisotropic diffusion, higher-order models involving curvature). Eventually, we discuss their numerical solution (steepest descent, iterative methods, dual solvers).

Relevant Courses

Useful: Functional Analysis, variational calculus, partial differential equations, numerical analysis.

References

[1] G. Aubert, and P. Kornprobst, *Mathematical Problems in Image Processing. Partial Differential Equations and the Calculus of Variations*, Springer, Applied Mathematical Sciences, Vol 147, (2006).

[2] T. F. Chan, and J. J. Shen, *Image Processing and Analysis - Variational, PDE, wavelet, and stochastic methods.* SIAM, (2005).

[3] Y. Meyer, *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, AMS 2001.

[4] L. Rudin, S. Osher, E. Fatemi, *Nonlinear Total Variation Based Noise Removal Algorithms*, Physica D: Nonlinear Phenomena, Vol. 60, Issues 1-4, 1992.

The Unified Method for PDEs and Medical Imaging (L16)

Thanasis Fokas

This course will discuss a new powerful method for analyzing linear boundary value problems. This method, often referred to as the "unified method", is based on the "synthesis" as opposed to separation of variables.

It has led to the analytical solution of several non-separable, as well as non-self adjoint boundary value problems. Furthermore, it has led to new numerical techniques for solving linear elliptic PDEs in the interior as well as in the exterior of polygons. The analytical and numerical implementation of the unified method to both evolution and elliptic PDEs will be discussed.

A related topic is the emergence of new analytical methods for solving inverse problems arising in medicine, including techniques for PET (Positron Emission Tomography) and SPECT (Single Photon Emission Computerized Tomography). A summary of these developments will also be presented.

Contemporary sampling techniques and compressed sensing (L16)

Non-Examinable (Graduate Level)

Anders Hansen

This is a graduate course on sampling theory and compressed sensing for use in signal processing and medical imaging. Compressed sensing is a theory of randomisation, sparsity and non-linear optimisation techniques that breaks traditional barriers in sampling theory. Since its introduction in 2004 the field has exploded and is rapidly growing and changing. Thus, we will take the word contemporary quite literally and emphasise the latest developments, however, no previous knowledge of the field is assumed. Although the main focus will be on compressed sensing, it will be presented in the general framework of sampling theory. The course will also present related areas of sampling theory such as generalised sampling.

The course will be fairly self contained, and applications will be emphasised (in particular, signal processing, Magnetic Resonance Imaging (MRI) and X-ray Tomography). The lectures will cover the most up to date research, and although this is a Part III course, it is also aimed at Phd students and post docs who are interested in using compressed sensing and generalised sampling in their research. Students from other disciplines than mathematics are encouraged to participate.

Desirable Previous Knowledge

Sampling theory and compressed sensing require a mix of mathematical tools from approximation theory, harmonic analysis, linear algebra, functional analysis, optimisation and probability theory. The course will contain discussions of both finite-dimensional and infinite-dimensional/analog signal models and thus linear algebra, Fourier analysis and functional analysis (at least basic Hilbert space theory) are important. The course will be self-contained, but students are encouraged to refresh their memories on properties of the Fourier transform as well as basic Hilbert space theory. Some basic knowledge of wavelets is useful as well as basic probability.

Introductory Reading

For a quick and dense review of basic Fourier analysis and functional analysis chapters 5 and 8 of "Real Analysis" (Folland) are good choices. For an introductory exposition to Hilbert space theory one may use "An Introduction to Hilbert Space" (Young). And for a review of wavelets see chapters 1 and 2 of "A First Course on Wavelets" (Hernandez, Weiss). The course will cover some of the chapters of "Compressed Sensing" (Eldar, Kutyniok), so to get a feeling about the topic one may consult chapter 1 as a start.

(a) Eldar, Y and Kutyniok, G., Compressed Sensing, CUP
(b) Folland, G. B., Real Analysis, Wiley.
(c) Hernandez, E. and Weiss, G., A First Course on Wavelets, CRC
(d) Young, N., An Introduction to Hilbert Space, CUP

Reading to complement course material

(a) Adcock, B and Hansen, A., Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon, Appl. Comp. Harm. Anal., 32 (2012)
(b) Candès, E. and Romberg, J. and Tao, T., Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory 52 (2006)
(c) Donoho, D., Compressed sensing, IEEE Trans. Inform. Theory 52 (2006)
(d) Körner, T. W., Fourier Analysis, CUP
(e) Reed, M. and Simon, B., Functional Analysis, Elsevier

Continuum Mechanics

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice familiarity with the continuum assumption and the material derivative will be assumed, as will basic ideas concerning incompressible, inviscid fluids mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable.

For solid mechanics courses no previous knowledge of solid mechanics is required, but prior knowledge of some continuum mechanics (e.g. an introductory course in fluid dynamics) will be assumed.

For both fluid dynamics and solid mechanics courses previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is highly desirable.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses.

Year	*Courses*
First	Differential Equations, Dynamics and Relativity, Vector Calculus, Vectors & Matrices.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves, Asymptotic Methods.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses, which may be found on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Fluid dynamics of the environment (M24)

C. P. Caulfield

Understanding, predicting and minimizing the impact of human activity on the environment is a central challenge for sustainability. Many of the key issues are associated with fluid motions in the ocean and the atmosphere, and this course provides an introduction to the basic fluid dynamics necessary to build mathematical models of the environment in which we live, focussing on flows which occur over sufficiently small time and length scales to be largely unaffected by the earth's rotation.

The course begins by considering the governing equations of fluid flow in the presence of (typically relatively small) density variation. When there are density variations in a fluid, it is possible for 'internal gravity waves' to occur, since the density variations within the fluid provide the restoring force, and the course will highlight some of the rich and surprising dynamics of these waves. In particular, internal gravity waves radiate energy vertically as well as horizontally, and their interaction with boundaries can focus this energy and cause mixing far from where the energy was input. The subtle dynamics of stratified mixing by turbulence is then introduced through an exploration of some of its basic characteristics including the complex interplay between kinetic and potential energy in a sheared, density stratified flow which can lead to a wide variety of flow instabilities.

Of course, density variations can also drive the flow, and the course will consider two particularly important and related classes of such flows. First, a relatively localised source can drive the rise of a turbulent 'plume' of buoyant fluid. Volcanic eruption clouds and accidental releases of pollution are just two examples of such plumes. Second, when there are lateral gradients in fluid density interacting with horizontal or sloping boundaries, turbulent 'density currents' can develop. Sea breezes, avalanches, turbidity currents (where the density differences come from suspended sediments) and volcanic 'pyroclastic flows' are all examples of such density currents. The interaction of plumes and density currents with each other and with a stratified environment, such as the atmosphere and the ocean, will also be discussed.

Desirable Previous Knowledge

Undergraduate fluid dynamics.

Reading to complement course material

(a) B. R. Sutherland, Internal gravity waves, Cambridge University Press (2010).
(b) J. S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

Biological Physics (M24)

R.E. Goldstein & U. Keyser

This course will provide an overview of the physics and mathematical description of living systems. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology. The topics to be covered will span the range of length scales from molecular to ecological, with emphasis on key paradigms. Introductory material on statistical mechanics will provide background for much of the course. The subsequent topics will include *Microscopic Physics* – van der Waals forces, screened electrostatics, Borwnian motion, fluctuation-dissipation theorem; *Fluctuation-Induced Forces* – polymer physics, random walks, entropic forces, stiff chains, self-avoidance, dynamics, protein folding; *Elasticity* – differential geometry of curves and suraces, linear elasticity theory, thin plates and rods, Helfrich model for membranes, elastohydrodynamics; *Chemical Kinetics and Pattern Formation*– Michaelis-Menten kinetics, oscillations, excitable media, ion channels, action potentials, reaction-diffusion dynamics, Fitzhugh-Nagumo model, spiral waves; *Dynamics*– life at low Reynolds numbers, chemoreception, advection-diffusion problems.

Desirable Previous Knowledge

Some familiarity with statistical physics will be helpful.

Introductory Reading

(a) P. Nelson. *Biological Physics.* W.H. Freeman (2007).
(b) J.D. Murray. *Mathematical Biology I. & II.* Springer (2007, 2008).
(c) K. Dill & S. Bromberg. *Molecular Driving Forces.* Garland (2009).

Reading to complement course material

(a) B. Alberts, A. Johnson, J. Lewis, M. Raff, K. Roberts and P. Walter. *Molecular Biology of the Cell.* 5th edition. Garland Science (2007).
(b) J.N. Israelachvili. *Intermolecular and Surface Forces.* 2nd edition. Academic Press (1992).
(c) E.J.W. Verwey and J.Th.G. Overbeek. *Theory of the Stability of Lyophobic Colloids.* Elsevier (1948).

(d) M. Doi and S.F. Edwards. *The Theory of Polymer Dynamics.* OUP (1986).
(e) A. Parsegian. *Van der Waals Forces.* CUP (2005).
(f) D. Andelman & W. Poon. *Condensed Matter Physics in Molecular and Cell Biology.* Taylor & Francis (2006).

Direct and Inverse Scattering of Waves (L16)

Orsola Rath-Spivack

The study of wave scattering is concerned with how the propagation of waves is affected by objects, and has a variety of applications in many fields, from environmental science to seismology, medicine, telecommunications, materials science, military applications, and many others.

If we know the nature of the objects and we want to find how an incident wave is scattered, we call this a 'direct scattering problem' and practical applications will include for example underwater sound propagation, light transmission through the atmosphere, or the effect of noise in built-up areas. If we measure and know the scattered field produced by an incident wave, but we do not know the nature of the objects that have scattered it, we call this an 'inverse scattering problem' and applications will include for example non-destructive tesing of materials, remote sensing with radar or lidar, or medical imaging.

This course will provide the basic theory of wave propagation and scattering and an overview of the main mathematical methods and approximations, with particular emphasis on inhomogeneous and random media, and on the regularisation of inverse scattering problems. Only time-harmonic waves will nbe normally considered.

Topics covered will include: 1. Boundary value problems and the integral form of the wave equation.

2. The parabolic equation and Born and Rytov approximations for the scattering problem.

3. Scattering by randomly rough surfaces and propagation in inhomogeneous media.

4. Ill-posedness of the inverse scattering problem, and the Moore-Penrose generalised inverse.

5. Regularisation methods and methods for solving some inverse scattering problems.

6. Time reversal and focusing in inhomogeneous media.

The lectures will be supplemented by example sheets and example classes.

Desirable Previous Knowledge

This course assumes basic knowledge of ODEs and PDEs, and of Fourier transforms. Some familiarity with linear algebra and with basic concepts in functional analysis is helpful, though by no means necessary.

Introductory Reading

For example:

(a) Landau, LD and Lifschitz, EM. *Fluid Mechanics*, Butterworth-Heinemann. [Chapter 8]
(b) Groetsch, CW *Inverse Problems in the Mathematical Sciences*, Braunschweig 1993

Reading to complement course material

(a) Crighton, DG et al. *Modern Methods in Analytical Acoustics*, ASA 1989.
(b) Jones, JD. *Acoustic and Electromagnetic Waves* Clarendon Press 1986
(c) Ishimaru, A *Wave Propagation and Scattering in Random Media*, Academic Press, 1978.
(d) Colton, D and Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer Verlag 1992.

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media will be analysed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Desirable Previous Knowledge

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Introductory Reading

(a) D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7

(b) G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.

(c) L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 and 5.

Reading to complement course material

(a) J. Happel and H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).

(b) S. Kim and J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)

(c) C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).

(d) O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Computational Methods in Fluid Mechanics (M16)

Non-Examinable (Graduate Level)

E.J. Hinch

The aim of this course is to provide an overview of some of the computational methods used to solve the partial differential equations that arise in fluid dynamics and related fields. The idea is to *provide a feel for the computational methods rather than study them in depth (cf. the complementary aim of the course on the Numerical Analysis of Differential Equations). Although the course is non-examinable, project-type essays will be set on some of the material.*

The course will start with a four-lecture introduction to the numerical solution of the Navier-Stokes equations at moderate Reynolds number; the issues and difficulties will be highlighted.

Next some general issues will be covered in greater detail.

- Discretisations: finite difference, finite element and spectral.
- Time-Stepping: explicit, implicit, multi-step, splitting, symplectic.
- Solution of Linear Systems: dense solvers, structured matrices, iterative methods: multigrid, conjugate gradient, GMRES and alternatives, preconditioning, sparse direct methods, eigen-solvers, pseudo-spectra.

The remaining lectures will focus on specific issues selected from the following.

- Demonstration of the commercial software
- Implementation issues: code design, testing, data prefetch, cache issues, use of black-box routines, modular code design, language compliance.
- Methods for hyperbolic systems of equations such as the compressible Euler equations.
- Representation of surfaces: splines for curves, diffuse interface method, indicator functions in Volume of Fluid methods, level sets.
- Boundary Integral/Element Method.
- Fast Multipole Method.
- Parameter continuation.
- Lattice-Boltzmann and similar methods.

Desirable Previous Knowledge

Attendance at an introductory course in Numerical Analysis that has covered (at an elementary level) the solution of ordinary differential equations and linear systems will be assumed. Some familiarity with the Navier-Stokes equations and basic fluid phenomena will be helpful (as covered by a first course in Fluid Dynamics).

Reading to complement course material

(a) Boyd, J.P. (2000) *Chebyshev and Fourier Spectral Methods* Dover.

(b) Acton, F.S. (1990) *Numerical Methods That Work* Mathematical Association of America.

(c) LeVeque, R.J. (1992) *Numerical Methods for Conservation Laws* Birkhauser-Verlag.

(d) Saad, Y. (1996) *Iterative Methods for Sparse Linear Systems* PWS.

(e) Barrett, R. et al. (1994) *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods* SIAM.

(f) Iserles, A. (1996) *A First Course in the Numerical Analysis of Differential Equations* CUP.

Perturbation and Stability Methods (M24)

J.M. Rallison & S.J. Cowley

The first part of this course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of the most useful mathematical tools for research will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

The second part of the course covers applications of perturbation methods to the study of fluid flows. So-called 'hydrodynamic stability' is a very broad discipline, and in this course we will concentrate on the stability of nearly parallel-flows (as for example arise in boundary-layer flows).

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. [6]
- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKBJLG' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium, including investigation of the rescaling required near 'hot spots', or'caustics'). [5]
- *The Summation of Series.* Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, Domb-Sykes plots. [1]
- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. At the end of this section further examples will be given of asymptotics beyond all orders. [6]
- *Stability Theory.* This section will review both eigenvalue and 'non-eigenvalue' aspects of stability theory as applied to fluid flows, concentrating on nearly-parallel flows. Aspects that will be covered include the concepts of 'causality' and the Briggs-Bers technique, the continuous spectrum, and the transitory algebraic growth that can follow from the fact that the operators in hydrodynamic stability theory are often not self-adjoint. [6]

In addition to the lectures, a series of examples sheets will be provided. The lecturers will run examples classes in parallel to the course.

Desirable Previous Knowledge

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve simple differential equations and partial differential equations and evaluate simple integrals.

Introductory Reading

(a) E.J. Hinch. *Perturbation Methods*, Cambridge University Press (1991).

(b) M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975).

Reading to Complement Course Material

(a) C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *Beware: Bender and Orszag call Stokes lines anti-Stokes lines, and vice versa.*

(b) John P. Boyd. *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series* Acta Applicandae, **56**, 1-98 (1999), and also available at
http://hdl.handle.net/2027.42/41670 and
http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf

(c) M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).

(d) P.G. Drazin and W. H. Reid. *Hydrodynamic Stability*, Cambridge University Press (1981 and 2004).

(e) J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer (1981).

(f) Peter Schmid and Dan S. Henningson. *Stability and Transition in Shear Flows*, Springer-Verlag (2001).

Sound Generation and Propagation (L16)

E.J. Brambley

The application of wave theory to problems involving the generation, propagation and scattering of acoustic and other waves is of considerable relevance in many practical situations. These include, for example, underwater sound propagation, aircraft noise, remote sensing, the effect of noise in built-up areas, and a variety of medical diagnostic applications. This course aims to provide the basic theory of wave generation, propagation and scattering, and an overview of the mathematical methods and approximations used to tackle these problems, with emphasis on applications to aeroacoustics. The course will cover some general aeroacoustic theory [3], sound generation by turbulence and moving bodies (including the Lighthill and Ffowcs Williams–Hawkings acoustic analogies) [3], scattering (including the scalar Wiener-Hopf technique applied to the Sommerfeld problem of scattering by a sharp edge) [4], long-distance sound propagation including nonlinear and viscous effects [3], and wave-guides [3]. The lectures will be supplemented by three examples sheets and examples classes.

Students considering this course might also like to consider the complementary course "Direct and Inverse Scattering of Waves".

Desirable Previous Knowledge

This course assumes that students have attended some introductory courses in continuum mechanics and complex variable theory (especially Fourier transforms and their inversion using of complex residues). Attendance at the Part III course Perturbation and Stability Methods would also be helpful, but is by no means essential.

Introductory Reading

(a) Landau, LD and Lifschitz, EM. Fluid Mechanics, Butterworth-Heinemann. [Chapters 1 & 8]

(b) Hinch, EJ. Perturbation Methods, CUP. [Chapters 3 & 7]

Reading to complement course material

(a) Crighton, DG et al. Modern Methods in Analytical Acoustics, ASA.

(b) Pierce, AD. Acoustics, McGraw–Hill.

(c) Noble, B. Methods based on the Wiener–Hopf technique, Chelsea. [Chapter 1]

Convection (L16)

Prof. M.R.E.Proctor

Convection is the name given to the means used by fluids to transfer heat when fluid flow is more effective than conduction. In a fluid layer, for example, between horizontal boundaries held at fixed temperatures, convection occurs when the temperature difference is sufficiently large. The onset of convection can be thought of as an instability of pattern forming type, and there are many interesting questions that can be asked: What are the pattern and horizontal scale of convection near onset? How does the heat transfer depend on the temperature difference? How do the simple patterns seen at onset break down? What is the effect on convection of other physical effects such as rotation, and what happens when there are two sources of buoyancy, such as thermohaline convection?

The course will address many of these issues. Though convection requires that fluid density depend on temperature and so be non-uniform, most of the course will use the Boussinesq approximation, in which the fluid may be treated as incompressible except for the buoyancy term. This approximation is a good one for laboratory liquids and gives a good guide to many aspect of convection for which the approximation is not accurate.

There will be three problem sheets and associated examples classes.

Desirable Previous Knowledge

Knowledge of fluid dynamics and dynamical systems would be an advantage.

Introductory Reading

(a) Chandrasekhar, S. Hydrodynamic and Hydromagnetic Stability. Dover

(b) Drazin, P and Read, W. Hydromagnetic stability (chapter 2). CUP

Reading to complement course material

(a) Getling, A.V. Rayleigh-Benard convection: structures and dynamics. World Scientific

(b) Hoyle, R. Pattern Formation. CUP

Complex and Biological Fluids (L24)

Eric Lauga

Fluid mechanics plays a crucial role in a number of biological processes, from the largest of animals to the smallest of cells. In this course, we will give an overview of the hydrodynamic phenomena associated with biological life at the cellular scale, from the fluid mechanics of individual microorganisms and their appendages to the modelling of collective cell dynamics. We will combine physical description, scaling analysis, and detailed calculations in order to present a wide overview of the subject, and appeal to students in applied mathematics, physics, and quantitative biology.

In the first part of the course, we will review the fluid dynamics and soft matter mechanics relevant to the locomotion of individual cells. Drawing examples from a variety of organisms, we will aim at providing a precise mathematical description of how cells actuate and exploit surrounding fluids in order to self-propel, and how they interact with their environment. The second part of the course will build on classical models for the flow of suspensions and polymeric fluids to derive the continuum framework describing the dynamics of populations of interacting cells. At the end of the course, students will be equipped to carry out independent research in biological physics and mechanics relevant to the cellular world. The lectures will be accompanied by example sheets and example classes.

Desirable Previous Knowledge

Undergraduate fluid dynamics, some knowledge of vector calculus and mathematical methods.

Introductory Reading

(a) Purcell (1977) Life at low Reynolds number. *American Journal of Physics*, **45**, 3-11.

(b) Yates (1986) How microorganisms move through water. *American Scientist* **74**, 358-365.

(c) Vogel (1996) Life in Moving Fluids, Princeton University Press.

(d) Berg (2000) Motile Behavior of Bacteria. *Physics Today*, **53**, 24.

(e) National Committee for Fluid Mechanics Films on "Rheological Behavior of Fluids" and "Low Reynolds Number Flow" at: `http://web.mit.edu/hml/ncfmf.html`

Reading to complement course material

(a) Lighthill (1975) Mathematical Biofluiddynamics, SIAM.

(b) Childress (1981) Mechanics of Flying and Swimming, CUP.

(c) Bird, Curtiss, Armstrong, and Hassager (1987) Dynamics of Polymeric Liquids, Vol. 1: Fluid Mechanics, 2nd ed, Wiley.

(d) Bray (2000) Cell Movements, Garland.

(e) Morrison (2001) Understanding Rheology, OUP.

Fluid Dynamics of Energy Systems. (L16)

J. Neufeld and A. Woods

This course will be divided into two main sections. Fist, it will explore some of the fluid dynamics involved in the energy supply sector, including oil, gas and geothermal energy, as well as a brief discussion of wind and tidal energy systems. Then it will examine some of the fluid mechanical challenges for efficient use of energy in buildings, especially through use of natural ventilation.

This will include a description of the fluid dynamics of oil and gas reservoir formation, including the formation of large sedimentary deposits from particle laden flows on the sea floor and their subsequent burial and compaction, followed by the natural migration of oil from source rock into reservoir rocks. The course will also examine the subsequent displacement of oil and gas such permeable rocks, either through primary pressure driven flow or as it is displaced by waterm , and reactive chemical solutions, injected into system. The emerging area of carbon sequestration will also be discussed, illustrating the dynamics controlling the dispersal of large volumes of CO2 in subsurface aquifers, and the longer term migration controlled by buoyancy forces. [8 lectures]

The fluid dynamics involved in the production of geothermal energy will also be discussed, illustrating how thermal energy can be transported through permeable rocks by the controlled flow of water and vapour. This will include a discussion of the phase changes involved in superheated systems, and of the deposition and dissolution of minerals as the temperature of the fluids migrating through such systems change. [2 lectures]

The course will then dscribe some of the fluid mechanical challenges for wind turbines and tidal turbines, especially related to efficiency of power generation and dynamics of the wakes [2 lectures].

Some of the challenges of energy efficiency will also be presented, including the use of natural flows in buildings to reduce the enormous energy demand of air-conditioning systems. Descriptions of the interaction between wind and buoyancy driven air flows, with heat exchange to the mass of a building will be discussed, as well as the detailed flow patterns within a building arising from localised and distributed sources of heating or cooling, which lead to turbulent plumes mixing the interior of confined spaces.[6 lectures]

Desirable Previous Knowledge

Part 1B and Part II Fluid Mechanics, and knowledge of partial differential equations.

Introductory Reading

(a) OM Phillips, Flow and reactions in permeable rock. CUP 1991

(a) JS Turner, Buoyancy effects in fluids, CUP, 1979.

Demonstrations in Fluid Mechanics. (L8)

Non-Examinable (Graduate Level)

J. Neufeld and G. Worster

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

Desirable Previous Knowledge

Undergraduate Fluid Dynamics.

Reading to complement course material

(a) M. Van Dyke. An Album of Fluid Motion. Parabolic Press.

(b) G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.

(c) M. Samimy, K. Breuer, P. Steen, and L. G. Leal. A Gallery of Fluid Motion. CUP.