Ordinary and Modular Representations of Finite Symmetric Groups

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Introduction
The symmetric groups, \( S_n \), are probably the most important class of finite groups due to Cayley’s theorem (Every group \( G \) is isomorphic to a subgroup of a symmetric group). The ordinary representation theory of \( S_n \) is relatively well understood, but the modular theory, started by Dickson in 1902 and advanced massively by Brauer in 1935, is still under development. Indeed elementary quantities such as the dimension of the irreducible \( p \)-modular representations are still widely unknown.

On this poster, we shall construct the ordinary and modular representations for the finite symmetric group. In ordinary representation theory, we can explicitly describe how the ordinary permutation modules decompose into the irreducible ones and we can begin to consider how these are related to the \( p \)-modular representation. This brings us the very rich research area of decomposition matrices. These show the factors of the irreducible ordinary representations on reduction modulo \( p \).

Definitions
The collection of all permutations of \( \{1, 2, \ldots, n\} \) forms a group under the composition of functions. We call this group \( S_n \) and refer to it as the symmetric group.

We let \( p \to 1 \in \mathbb{Z} \) be a prime. For every group homomorphism \( \rho : G \to GL(V) \) we say that \( V \) is a \( \rho \)-representation of \( G \). Indeed, \( \rho \) is a group homomorphism: \( \rho(1) = id_V \). Ordinary and Modular Representations of Finite Symmetric Groups

Constructing Representations
We can now define the module \( M^\rho \) as the vector space over a field \( F \) with basis consisting of the \( \rho \)-tableaux, which are defined as the collection of \( n \)-element permutations of \( S_n \). Let \( \pi, \tau \in S_n \). Then \( \pi \tau \) is defined as \( \{i \in [n] : \pi(i) < \pi(i) \} \). The \( \rho \)-tableaux are then given by \( \rho(\pi) \).

Alternatively, this can be thought of as the permutation module of \( S_n \), acting on the set of the Young subgroup \( S_n \). \( S_n \) is the collection of \( \{\pi \in S_n : \pi \cdot \pi = \pi \} \). In most cases this is not irreducible, indeed the subspace generated by the sum of the \( \rho \)-tableaux is a one-dimensional subspace, on which \( S_n \) acts trivially.

We now seek a particular type of submodular, called the Specht module. Define \( \nu \) to be \( \{1, 2, \ldots, n\} \) to be the set of all permutations of \( S_n \). Then \( \nu \) is the orthogonal complement with respect to the bilinear form \( s_{\lambda}^* s_{\mu} = \sum_{\pi \in \nu} s_{\lambda} \pi s_{\mu} \).

To deal with irreducibility, we require the Submodular Theorem: If \( V \) is a submodule of \( M^\rho \), the either \( \nu \to \{0\} \) or \( \nu \to S_n \), and \( \nu \to S_n \) is the orthogonal complement with respect to the bilinear form \( s_{\lambda}^* s_{\mu} = \sum_{\pi \in \nu} s_{\lambda} \pi s_{\mu} \).

In the ordinary case, \( \nu \to \{0\} \) is an inner product to \( S_n \) and the only \( n \)-element permutations of \( S_n \) are irreducible. It can be shown that \( S_n \) is irreducible. It can be shown that \( S_n \) is the only irreducible module for \( S_n \). However, with other permutation modules, \( M^\rho \) and \( S_n \) depend only on the prime subfield (the subfield generated by \( \pi \)). For the characteristic zero case, it suffices to only consider the field \( \mathbb{Q} \). In this case, the decomposition of \( M^\rho \) into the Specht modules \( S_n \) is given by Young’s rule: the multiplicity of \( S_n \) as a factor of \( M^\rho \) is the number of semistandard \( \lambda \)-tableaux of \( \rho \), where a tableau has type \( \pi \) if \( \rho(\pi) \) if the tableau contains \( \lambda \). It is semistandard if the numbers strictly increase down the columns and are non-decreasing along the rows. This reduces the complex algebraic questions of decomposition into a simple case of counting.

As an example, we shall calculate the multiplicity of \( S_n \) as a factor of \( M^\rho \). By Young’s rule, we seek semistandard \( \lambda \)-tableaux of \( \{1, 2, \ldots, n\} \), where the tableau contains \( \lambda \) if \( \rho(\pi) \).

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Modular Representations
Now we understand the ordinary representations, we can now change focus to the modular representations. An advanced result within modular representation theory states that the number of irreducible modular representations is equal to the number of conjugacy classes whose elements have order coprime to \( char(F) \). For \( S_n \), this is equal to the number of partitions whose components are all coprime to \( p \) (the order of an element with cycle type \( \{p_1, p_2, \ldots, p_k\} \) is the least common multiple of \( p_1, p_2, \ldots, p_k \)). This in turn is equal to the number of partitions of \( n \) in which no component is repeated \( p \) times, called the \( p \)-regular partitions. Otherwise we say \( \rho \) is \( p \)-singular. Conversely, it is precisely for the \( p \)-regular partitions that \( D^p = 1 \) is non-zero. This gives all the irreducible \( p \)-modular representations for \( S_n \).

Decomposition Matrices
A decomposition matrix records the multiplicities of the irreducible \( p \)-modular representations \( D^p \) in the reductions of the irreducible ordinary representations \( S_n \). Explicitly the rows are parameterised by the \( S_n \) and the columns by the \( D^p \), and the entry at \( [S_n, D^p] \) is the multiplicity of \( D^p \) as a factor of the reduction module \( p \) of \( S_n \). The previous discussion shows, for \( p \) a regular, \( [S_n, D^p] \) is non-zero, indeed this factor is unique, as \( [S_n, D^p] = 1 \).

Given partitions \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), we define a partial order by \( \mu \triangleright \lambda \) if and only if \( \sum_{\mu_i > 1} \mu_i \geq \sum_{\lambda_i > 1} \lambda_i \). From the \( \lambda = (1, 1, 2, \ldots, 2) \). One can show that \( D^p \) is a factor of \( S_n \) only if \( p \triangleright \lambda \). This forces the decomposition matrix to have the following shape:

\[
\begin{pmatrix}
1 & 0 & \cdots & 1 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0
\end{pmatrix}
\]

Where we have ordered the columns according to the partial order and the rows such that all the \( p \)-regular partitions occur before the \( p \)-singular partitions.

As of yet, there is no general algorithm for calculating these matrices, however there are many partial results for particular characteristics and partitions. The following theorem gives a flavour of the results currently known and constructs the section of the decomposition matrix corresponding to the partitions of the form \( (n, n^{2-1}) \), called the hook partitions.

Theorem: Suppose \( p \) is odd.

1. \( p \)-divides \( n \), all the hook representation of \( S_n \) is irreducible as \( p \)-modular representations and no two are isomorphic.
2. \( p \)-divides \( n \), part of the decomposition matrix of \( S_n \) is

\[
\begin{pmatrix}
(n) & 0 & 1 \\
(n-1, 1) & 1 & 1 \\
(n-2, 1^2) & 1 & 1 \\
\vdots & \ddots & \ddots \\
(1) & \cdots & 1 \\
(1^{n-1}) & 1 & 1 \\
0 & 1
\end{pmatrix}
\]

For the interested reader, I point them to “The Representation Theory of the Symmetric Groups” by G. D. James which contains thorough proofs of the information on this poster and many other results in this field.