UNIVERSITY OF CAMBRIDGE

MAMA/359, NST3AS/359, MAAS/359

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 2:00pm to 5:00pm

PAPER 359

MATHEMATICAL ANALYSIS OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The first question carries 60 marks and the second question 40 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 You may feel free to use, without proof, standard Sobolev embedding and interpolation theorems.

Let $\Omega = \mathbb{T}^3 := (\mathbb{R}/L\mathbb{Z})^3$ be a fundamental periodic domain in $\mathbb{R}^3$, and let

$$H = \left\{ v \in (\dot{L}^2_{\text{per}}(\Omega))^3 \,\middle|\, \nabla \cdot v = 0 \right\},$$

with $|\cdot|$ denoting the norm of the space H. Recall the Leray-Helmholtz orthogonal projection

$$P_\sigma : (\dot{L}^2_{\text{per}}(\Omega))^3 \to H,$$

and denote by $D(A) = (\dot{H}^2_{\text{per}}(\Omega))^3 \cap H$ the domain of the Stokes operator A, where $A = -P_\sigma \Delta = -\Delta$ in this setting.

Suppose that $\{\varphi_1, \varphi_2, \dots\}$ is an orthonormal basis of the Hilbert space H consisting of eigenfunctions of the Stokes operator A, that is $A\varphi_k = \lambda_k \varphi_k$ for $k = 1, 2, \dots$, with $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$, repeated according to their multiplicities. Moreover, suppose that $\{\psi_1, \psi_2, \dots\}$ is an orthonormal basis of the Hilbert space $\dot{L}^2_{\text{per}}(\Omega)$ consisting of eigenfunctions of the Laplacian operator $-\Delta$, that is $-\Delta\psi_k = \mu_k \psi_k$ for $k = 1, 2, \dots$, with $0 < \mu_1 \leqslant \mu_2 \leqslant \dots$, repeated according to their multiplicities. Furthermore, let

$$H_n = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\} \quad \text{and} \quad \tilde{H}_n = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_n\},$$

for $n = 1, 2, \dots$, and denote by P_n the orthogonal projection $P_n : H \to H_n$, and by Π_n the orthogonal projection $\Pi_n : \dot{L}^2_{\text{per}}(\Omega) \to \tilde{H}_n$.

Consider the following Rayleight-Bénard convection system of coupled PDEs in $\mathbb{R}^3$, subject to periodic boundary conditions with fundamental domain Ω,

$$\begin{aligned}
&-\nu\Delta u + \nabla p = \alpha\theta e_3, \\
&\nabla \cdot u = 0, \\
&\frac{\partial}{\partial t}\theta - \kappa\Delta\theta + (u \cdot \nabla)\theta = \beta(u \cdot e_3), \\
&\theta(x, 0) = \theta_0(x),
\end{aligned}$$

where the unknowns are the three-dimensional velocity vector field u, the pressure p and the temperature θ, all are functions of (x, t) and periodic with respect to the spatial variable $x \in \mathbb{R}^3$. Moreover, ν, α, β and κ are given positive constants, and $e_3 = (0, 0, 1)$ is the unit vector in the vertical direction. Or equivalently consider the system

$$\left.\begin{aligned}
&\nu A u = \alpha P_\sigma(\theta e_3) \\
&\frac{\partial}{\partial t}\theta - \kappa\Delta\theta + (u \cdot \nabla)\theta = \beta(u \cdot e_3)
\end{aligned}\right\} \qquad (*)$$

with initial value $\theta(x, 0) = \theta_0(x)$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(a) Show that for every $v \in D(A)$ and every $\eta, \chi \in \dot{H}^1_{\mathrm{per}}(\Omega)$, one has:

(i)
$$\|(v \cdot \nabla)\eta\|_{\dot{L}^2_{\mathrm{per}}(\Omega)} \leqslant c|Av|\|\eta\|_{\dot{H}^1_{\mathrm{per}}(\Omega)},$$
for some positive constant c,

(ii)
$$\int_\Omega ((v(x) \cdot \nabla)\eta(x))\,\chi(x)\,dx = -\int_\Omega ((v(x) \cdot \nabla)\chi(x))\,\eta(x)\,dx,$$
and

(iii)
$$\int_\Omega ((v(x) \cdot \nabla)\eta(x))\,\eta(x)\,dx = 0.$$

(b) Let $\theta_0 \in \dot{L}^2_{\mathrm{per}}$ be given, and consider the following Galerkin approximation scheme for system $(*)$:

$$\left.\begin{aligned} &\nu A u_m = \alpha P_m P_\sigma(\theta_m e_3) \\ &\frac{\partial}{\partial t}\theta_m - \kappa\Delta\theta_m + \Pi_m\left((u_m \cdot \nabla)\theta_m\right) = \beta\Pi_m(u_m \cdot e_3) \end{aligned}\right\} \qquad (**)$$

with initial value $\theta_m(0) = \Pi_m\theta_0$, and where $u_m \in H_m$ and $\theta_m \in \tilde{H}_m$.

(i) Show that for every $u_m \in H_m$ and $\theta_m \in \tilde{H}_m$ satisfying $(**)$ one has
$$|Au_m| \leqslant \frac{\alpha}{\nu}\,\|\theta_m\|_{\dot{L}^2_{\mathrm{per}}(\Omega)}.$$

(ii) Explain briefly why for every $T > 0$ system $(**)$ has a unique solution in the time interval $[0,T]$.

(iii) Demonstrate that for every $T > 0$, there are positive constants $K_0(T), K_2(T), \overline{K_1}(T)$ and $\overline{K_0'}(T)$, independent of m, such that the solution u_m and θ_m of the Galerkin system $(**)$ satisfies:
$$\|\theta_m\|_{L^\infty((0,T);\dot{L}^2_{\mathrm{per}}(\Omega))} \leqslant K_0(T), \quad \|Au_m\|_{L^\infty((0,T);H)} \leqslant K_2(T),$$
$$\|\theta_m\|_{L^2((0,T);\dot{H}^1_{\mathrm{per}}(\Omega))} \leqslant \overline{K_1}(T),$$
and
$$\left\|\frac{d\theta_m}{dt}\right\|_{L^2((0,T);H^{-1}_{\mathrm{per}}(\Omega))} \leqslant \overline{K_0'}(T),$$
where $H^{-1}_{\mathrm{per}}(\Omega)$ denotes the dual space of $\dot{H}^1_{\mathrm{per}}(\Omega)$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(c) Let us denote by $\dot{L}^2_{\text{weak}}$ the Hilbert space $\dot{L}^2_{\text{per}}(\Omega)$ endowed with its weak topology. Use the above to prove that for every $T > 0$ and every choice of initial data $\theta_0 \in \dot{L}^2_{\text{per}}(\Omega)$ there exists a global weak solution, θ and u to system $(*)$ satisfying the following properties:

(i)
$$\theta \in C([0,T]; \dot{L}^2_{\text{weak}}) \bigcap L^\infty((0,T); \dot{L}^2_{\text{per}}(\Omega)) \bigcap L^2((0,T); \dot{H}^1_{\text{per}}(\Omega)),$$
and
$$\frac{d\theta}{dt} \in L^2((0,T); H^{-1}_{\text{per}}(\Omega)).$$

(ii) The weak solution established above satisfies the first equation of system $(*)$ in $L^\infty((0,T); H)$ and the second equation of system $(*)$ in $L^2((0,T); H^{-1}_{\text{per}}(\Omega))$.

(d) Demonstrate that there is a unique weak solution to system $(*)$ satisfying the properties stated in (c)(i) and (c)(ii).

(e) Assume that the unique weak solution to system $(*)$, θ and u, which was established in (c) and (d) satisfies the energy equality:

$$\frac{1}{2}\|\theta(t)\|^2_{\dot{L}^2_{\text{per}}(\Omega)} + \kappa \int_0^t \|\nabla\theta(\tau)\|^2_{\dot{L}^2_{\text{per}}(\Omega)}\, d\tau = \frac{1}{2}\|\theta_0\|^2_{\dot{L}^2_{\text{per}}(\Omega)} + \beta \int_0^t \int_\Omega (u(x,t)\cdot e_3)\theta(x)\, dx\, dt,$$

for every $t \in [0,T]$.

Show that the unique weak solution established above satisfies the following additional regularity property
$$\theta \in C([0,T]; \dot{L}^2_{\text{per}}(\Omega)).$$

2 You may feel free to use, without proof, standard Sobolev embedding and interpolation theorems.

Let $\Omega = \mathbb{T}^2 := (\mathbb{R}/L\mathbb{Z})^2$ be a fundamental periodic domain in the plane $\mathbb{R}^2$, and let

$$H = \left\{ v \in (\dot{L}^2_{\mathrm{per}}(\Omega))^2 \middle| \quad \nabla \cdot v = 0 \right\}.$$

Suppose that $\{\psi_1, \psi_2, \cdots\}$ is an orthonormal basis of the Hilbert space $\dot{L}^2_{\mathrm{per}}(\Omega)$ consisting of eigenfunctions of the Laplacian operator $-\Delta$, with domain $D(-\Delta) = \dot{H}^2_{\mathrm{per}}(\Omega)$; that is $-\Delta\psi_k = \mu_k \psi_k$ for $k = 1, 2, \cdots$, with $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$, repeated according to their multiplicities. Furthermore, for $n = 1, 2, \ldots$, denote by

$$\tilde{H}_n = \mathrm{span}\{\psi_1, \psi_2, \cdots, \psi_n\} \quad \text{and} \quad H_n = \mathrm{span}\{\nabla^\perp \psi_1, \nabla^\perp \psi_2, \ldots, \nabla^\perp \psi_n\}$$

where the operator $\nabla^\perp := (\frac{\partial}{\partial y}, -\frac{\partial}{\partial x})$. Observe that $\tilde{H}_n$ and H_n consist of trigonometric polynomials. We also denote by $\tilde{P}_n$ the orthogonal projection $\tilde{P}_n : \dot{L}^2_{\mathrm{per}}(\Omega) \to \tilde{H}_n$.

Assumptions Assume that the following equalities (i)-(iv), below, hold true

(i) for every $\chi, \phi \in \dot{H}^2_{\mathrm{per}}(\Omega)$ and $j = 1, 2$ one has

$$\frac{\partial(\chi\phi)}{\partial x_j} = \frac{\partial \chi}{\partial x_j}\phi + \chi \frac{\partial \phi}{\partial x_j}, \quad \text{and the equality holds in} \quad \dot{H}^1_{\mathrm{per}}(\Omega),$$

(ii) for every $\chi, \phi \in \dot{H}^1_{\mathrm{per}}(\Omega)$ and $j = 1, 2$ one has

$$\int_\Omega \frac{\partial \chi}{\partial x_j} \phi \, dx = -\int_\Omega \chi \frac{\partial \phi}{\partial x_j} \, dx,$$

(iii) for every $\chi, \phi \in \dot{H}^2_{\mathrm{per}}(\Omega)$

$$\int_\Omega -\Delta\chi \, \phi \, dx = \int_\Omega \nabla\chi \cdot \nabla\phi \, dx.$$

(iv) For every $v \in (\dot{H}^{s+1}_{\mathrm{per}}(\Omega))^2 \bigcap H$ and $\xi = \nabla \times v := \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$, where $s \geqslant 0$, one has

$$\|v\|_{(\dot{H}^{s+1}_{\mathrm{per}})^2} = \|\xi\|_{\dot{H}^s_{\mathrm{per}}}.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

Consider the vorticity formulation of the two-dimensional steady-state damped-driven Navier-Stokes equations, subject to periodic boundary conditions with fundamental domain Ω,

$$\begin{aligned} &-\nu\Delta\omega + \gamma\omega + (u\cdot\nabla)\omega = g, \\ &u = \nabla^{\perp}\Psi := (\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}), \\ &-\Delta\Psi = \omega, \end{aligned} \tag{$\dagger$}$$

where the unknowns are the two-dimensional vector field u and the scalar functions, ω and Ψ, the vorticity and stream function, respectively. Moreover, ν and γ are given positive constants, and $g \in \dot{L}^2_{\text{per}}(\Omega)$ is a given forcing scalar function. From (†) one can easily see that

$$\nabla\cdot u = 0 \quad \text{and} \quad \nabla\times u = \omega\,.$$

(a) Consider the following Galerkin approximation scheme for system (†):

$$\begin{aligned} &-\nu\Delta\omega_m + \gamma\omega_m + \tilde{P}_m((u_m\cdot\nabla)\omega_m) = \tilde{P}_m g\,, \\ &u_m = (\frac{\partial\Psi_m}{\partial y}, -\frac{\partial\Psi_m}{\partial x}), \\ &-\Delta\Psi_m = \omega_m\,, \end{aligned} \tag{$\dagger\dagger$}$$

where $\omega_m, \Psi_m \in \tilde{H}_m$ and $u_m \in H_m$.

(i) Show that for every $m = 1, 2, \ldots$ all the solutions of (††) satisfy

$$\nu\,\|\nabla\omega_m\|^2_{\dot{L}^2_{\text{per}}}\,, \gamma\,\|\omega_m\|^2_{\dot{L}^2_{\text{per}}} \leqslant R_1^2\,,$$

for some constant $R_1 \geqslant 0$, which does not depend on m, but depends on $\|g\|_{\dot{L}^2_{\text{per}}}$ and γ . Moreover, show that

$$\nu\,\|\Delta\omega_m\|^2_{\dot{L}^2_{\text{per}}}\,, \gamma\,\|\nabla\omega_m\|^2_{\dot{L}^2_{\text{per}}} \leqslant R_2^2\,,$$

for some constant $R_2 \geqslant 0$, which does not depend on m, but depends on $\|g\|_{\dot{L}^2_{\text{per}}}\,, \gamma, \nu$ and μ_1 .

(ii) Prove that for every $m = 1, 2, \ldots$ the nonlinear system (††) has a solution $\omega_m, \Psi_m \in \tilde{H}_m$ and $u_m \in H_m$.
[Hint: You may want to consider the map $F_m : \tilde{H}_m \to \tilde{H}_m$ defined by

$$F_m(w) = -w + (-\nu\Delta)^{-1}\left(-\gamma w - \tilde{P}_m((v\cdot\nabla)w) + \tilde{P}_m g\right),$$

where $v = (\frac{\partial\Phi}{\partial y}, -\frac{\partial\Phi}{\partial x}) \in H_m$ and $-\Delta\Phi = w \in \tilde{H}_m$, and show, by quoting a lemma based on the Brouwer fixed point theorem, or otherwise, that there is $w^* \in \tilde{H}_m$ such that $F_m(w^*) = 0$].

[QUESTION CONTINUES ON THE NEXT PAGE]

(b) (i) Use part (a) to show that there exists $\omega \in \dot{H}^2_{\mathrm{per}}(\Omega)$, with

$$\nu \|\Delta\omega\|^2_{\dot{L}^2_{\mathrm{per}}}, \gamma \|\nabla\omega\|^2_{\dot{L}^2_{\mathrm{per}}} \leqslant R_2^2,$$

$\Psi \in \dot{H}^4_{\mathrm{per}}(\Omega)$ and $u \in \dot{H}^3_{\mathrm{per}}(\Omega) \bigcap H$ which satisfy (†) in a weak sense and in the appropriate spaces, and in particular that the first equation of (†) holds in $\dot{L}^2_{\mathrm{per}}(\Omega)$.

(ii) Moreover, show that the solution established in (i) satisfies the following equality in $\dot{L}^2_{\mathrm{per}}(\Omega)$

$$u \cdot \nabla\omega = \nabla \cdot (u\omega).$$

(c) Suppose now that $g \in \dot{L}^\infty_{\mathrm{per}}(\Omega)$. Demonstrate that the solution established in (b) to system (†) satisfies

$$\gamma \|\omega\|_{\dot{L}^\infty_{\mathrm{per}}} \leqslant \|g\|_{\dot{L}^\infty_{\mathrm{per}}}.$$

[Hint: You may want to consider multiplying equation (†) by ω^{2p+1}, where p is an arbitrary positive integer, and integrate over Ω to establish an appropriate bound for $\gamma \|\omega\|_{\dot{L}^{2p+2}_{\mathrm{per}}}$, and then take the limit as $p \to \infty$].

(d) Suppose again that $g \in \dot{L}^\infty_{\mathrm{per}}(\Omega)$. Consider the vorticity formulation of the two-dimensional steady-state damped-driven Euler's equations, subject to periodic boundary condition with fundamental domain Ω,

$$\begin{aligned} &\gamma\omega + \nabla \cdot (u\omega) = g, \\ &u = \nabla^\perp \Psi := (\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}), \\ &-\Delta\Psi = \omega. \end{aligned} \tag{‡}$$

Show that (‡) has a weak solution $\omega \in \dot{L}^\infty_{\mathrm{per}}(\Omega)$, $u \in \dot{H}^1_{\mathrm{per}}(\Omega) \bigcap H$ and $\Psi \in \dot{H}^2_{\mathrm{per}}(\Omega)$; moreover that the first equation in (‡) holds in $\dot{H}^{-1}_{\mathrm{per}}(\Omega)$, the dual of $\dot{H}^1_{\mathrm{per}}(\Omega)$.

[Hint: You may want to consider the limit as $\nu \to 0$ of the solutions to equation (†) established in parts (a)-(c)].

END OF PAPER