MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 10 June 2026 2:00pm to 4:00pm

PAPER 358

SPECTRAL COMPUTATIONS IN INFINITE DIMENSIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let A be a closed and densely defined operator on $\mathcal{H}$. Show that if there exists $z \in \mathbb{C}\backslash\mathrm{Sp}(A)$ such that $(A - zI)^{-1}$ is compact, then $(A - wI)^{-1}$ is compact for every $w \in \mathbb{C}\backslash\mathrm{Sp}(A)$. In this case, we say that A has *compact resolvent*. Show that if A has compact resolvent, then its spectrum consists of isolated points that are eigenvalues.

(b) The imaginary Airy operator is defined as

$$T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix, \quad \mathcal{D}(T) = \{f \in L^2(\mathbb{R}) : -f'' + ixf \in L^2(\mathbb{R})\}.$$

Alternatively, T is the closure of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix$ initially defined on smooth functions of compact support. Use integration by parts to show that

$$\mathcal{D}(T) = \{f \in H^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\},$$

where $H^2(\mathbb{R})$ is the Sobolev space of elements of $L^2(\mathbb{R})$ that have weak first and second derivatives in $L^2(\mathbb{R})$. Rellich's criterion implies that

$$\{f \in L^2(\mathbb{R}) : \|f\| \leqslant C, \|xf\| \leqslant C, \|f''\| \leqslant C\}$$

is a compact subset of $L^2(\mathbb{R})$ for each $C > 0$. Using this result, prove that T has compact resolvent and hence that its spectrum can only consist of isolated eigenvalues. Show, however, that $\|(T - zI)^{-1}\|^{-1}$ is constant along every vertical line in the complex plane. Conclude that $\mathrm{Sp}(T) = \emptyset$.

(c) Provide an example of a differential operator on $L^2(\mathbb{R}^2)$ with empty spectrum whose resolvent is not compact.

2 The *numerical range* of A is

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{D}(A), \|x\| = 1\}.$$

The *essential numerical range* of A is

$$W_e(A) = \bigcap_{B \text{ compact}} \mathrm{Cl}\left(W(A+B)\right).$$

It represents the portion of the numerical range that remains stable under compact perturbations of the operator, and it can be empty if A is unbounded. In this question you may use the following equivalent definition of $W_e(A)$:

$$W_e(A) = \left\{\lambda \in \mathbb{C} : \exists \{v_n\}_{n=1}^{\infty} \subset \mathcal{D}(A), \|v_n\| = 1, v_n \xrightarrow{w} 0, \lim_{n\to\infty} \langle Av_n, v_n\rangle = \lambda\right\},$$

where $v_n \xrightarrow{w} 0$ means that v_n converges weakly to zero.

(a) Let A be a bounded operator on a separable Hilbert space $\mathcal{H}$. Suppose that $\{\mathcal{P}_n\}$ is a sequence of orthogonal projections onto finite-dimensional subspaces such that $\mathcal{P}_n^*\mathcal{P}_n$ converges strongly to the identity, and such that $\lambda_n \in \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)$ have $\lambda_n \to \lambda \notin \mathrm{Sp}(A)$. Show that $\lambda \in W_e(A)$.

(b) Let $A \in \Omega_{\ell^2(\mathbb{N})}$ and $\mathcal{Q}_n$ be the orthogonal projection onto $(\mathrm{span}\{e_1, \ldots, e_n\})^\perp = \mathrm{Cl}(\mathrm{span}\{e_{n+1}, e_{n+2}, \ldots\})$. Show that if $W_e(A) \neq \emptyset$, then $\mathrm{Cl}(W(\mathcal{Q}_n A \mathcal{Q}_n^*)) \downarrow W_e(A)$ in the Attouch–Wets topology as $n \to \infty$. Show that if $W_e(A) = \emptyset$, then for each compact set K, $K \cap \mathrm{Cl}(W(\mathcal{Q}_n A \mathcal{Q}_n^*)) = \emptyset$ for sufficiently large n.

[QUESTION CONTINUES ON THE NEXT PAGE]

(c) Consider the non-self-adjoint (NSA) harmonic oscillator,

$$T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^2,$$

Recall the Hermite functions $\{\psi_m\}_{m=0}^{\infty}$ from lectures defined by

$$\psi_m(x) = \frac{1}{\sqrt{2^m m! \sqrt{\pi}}} e^{-x^2/2} H_m(x), \quad m \in \mathbb{Z}_{\geqslant 0},$$

which satisfy

$$\psi_m'(x) = \sqrt{\frac{m}{2}}\,\psi_{m-1}(x) - \sqrt{\frac{m+1}{2}}\,\psi_{m+1}(x),$$
$$x\psi_m(x) = \sqrt{\frac{m}{2}}\,\psi_{m-1}(x) + \sqrt{\frac{m+1}{2}}\,\psi_{m+1}(x),$$

for $m \in \mathbb{N}$. Let $\mathcal{Q}_n$ be the orthogonal projection onto $(\mathrm{span}\{\psi_0, \dots, \psi_n\})^{\perp}$ and $f = \sum_{m=n+1}^{\infty} c_m \psi_m$ be of Schwartz class and in the range of $\mathcal{Q}_n$ with $\|f\|_{L^2(\mathbb{R})}^2 = \sum_{m=n+1}^{\infty} |c_m|^2 = 1$. Show that

$$\langle Tf, f\rangle = (i-1)\|xf\|_{L^2(\mathbb{R})}^2 + \sum_{m=n+1}^{\infty} (2m+1)|c_m|^2.$$

Using this relation, show that if $|z| \leqslant n$, then $z \notin W(\mathcal{Q}_n T \mathcal{Q}_n^*)$. Conclude that $W_e(T) = \emptyset$.

3

(a) State the spectral theorem for normal operators on separable Hilbert spaces.

(b) Let A be a self-adjoint operator on $\mathcal{H}$. Prove that for $a, b \in \mathbb{R}$ with $a < b$ and $v \in \mathcal{H}$,

$$\lim_{\epsilon \downarrow 0} \left(\frac{1}{2\pi i} \int_a^b (A - (x + i\epsilon)I)^{-1} - (A - (x - i\epsilon)I)^{-1} \, \mathrm{d}x \right) v = \frac{1}{2} \big[\mathcal{E}((a,b)) + \mathcal{E}([a,b]) \big] v.$$

(c) Consider the operator

$$[Au](x) = (x^3 - x)u(x), \quad u \in L^2([-1,1]).$$

Given $f, g \in L^2([-1,1])$, define the scalar-valued spectral measure with respect to f and g:

$$\mu_{f,g}(S) = \langle \mathcal{E}(S) f, g \rangle.$$

Show that this measure is absolutely continuous and find its Radon–Nikodym derivative.

END OF PAPER