UNIVERSITY OF CAMBRIDGE

MAMA/357, NST3AS/357, MAAS/357

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 16 June 2026 2:00pm to 4:00pm

PAPER 357

GRAVITATIONAL WAVES AND NUMERICAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(\mathcal{M}, \boldsymbol{g})$ be a globally hyperbolic spacetime with Levi-Civita connextion $\Gamma^{\alpha}_{\beta\gamma}$ and Riemann tensor

$$R^{\lambda}{}_{\alpha\mu\beta} = \partial_{\mu}\Gamma^{\lambda}_{\alpha\beta} - \partial_{\beta}\Gamma^{\lambda}_{\alpha\mu} + \Gamma^{\kappa}_{\alpha\beta}\Gamma^{\lambda}_{\kappa\mu} - \Gamma^{\kappa}_{\alpha\mu}\Gamma^{\lambda}_{\kappa\beta}\,.$$

In adapted coordinates (t, x^i), the metric $g_{\alpha\beta}$ is related to the lapse function α, shift vector β^i and the spatial metric γ_{ij} of the 3+1 split by

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \quad g^{\alpha\beta} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}.$$

Here Greek indices range from 0 to 3 and Latin indices from 1 to 3. The determinants of the spacetime metric $g = \det g_{\alpha\beta}$ and the spatial metric $\gamma = \det \gamma_{ij}$ are related by $g = -\alpha^2\gamma$. They satisfy $\partial g/\partial g_{\alpha\beta} = g g^{\alpha\beta}$ and $\partial\gamma/\partial\gamma_{ij} = \gamma\gamma^{ij}$. A spacetime coordinate system x^{λ} is called *harmonic* if it satisfies the equation

$$\nabla^{\mu}\nabla_{\mu}x^{\lambda} = 0\,, \qquad (\dagger)$$

where the x^{λ} are treated as scalar functions in the covariant derivative.

(a) Briefly comment on whether Eq. (†) is a tensorial equation or not.

(b) Show that Eq. (†) can be formulated in the two equivalent forms

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \qquad \Leftrightarrow \qquad \partial_{\mu}\left(\sqrt{-g}\, g^{\lambda\mu}\right) = 0\,. \qquad (\ddagger)$$

(c) In adapted coordinates the 3+1 evolution equation for the spatial metric γ_{ij} is

$$\partial_t\gamma_{ij} = \beta^m\partial_m\gamma_{ij} + 2\gamma_{m(i}\partial_{j)}\beta^m - 2\alpha K_{ij}\,,$$

where K_{ij} is the extrinsic curvature. Show that this implies

$$\partial_t\gamma = \beta^m\partial_m\gamma + 2\gamma\partial_m\beta^m - 2\alpha\gamma K\,,$$

where $K := \gamma^{ij}K_{ij}$ is the trace of the extrinsic curvature.

(d) *Harmonic slicing* is defined by imposing Eq. (†) for $\lambda = 0$ only. Using the second relation of Eq. (‡), show that harmonic slicing belongs to the Bona-Massó family of slicing conditions,

$$(\partial_t - \beta^m\partial_m)\alpha = -\alpha^2 f(\alpha) K\,,$$

for a specific choice of the function $f(\alpha)$ which you should determine.

(e) Show that in harmonic gauge, i.e. Eq. (†) required for all $\lambda \in \{0, 1, 2, 3\}$, the vacuum Einstein equations $R_{\alpha\beta} = 0$ can be written in the form

$$R_{\alpha\beta} = -\frac{1}{2}g^{\mu\nu}\partial_{\mu}\partial_{\nu}g_{\alpha\beta} + \ldots = 0\,,$$

where the dots denote terms (which need not be evaluated explicitly) containing at most first derivatives of the metric components. Briefly interpret the principal part of this equation.

2 The Bondi metric in coordinates $x^\mu = (u, r, \theta, \phi)$ for an axisymmetric spacetime that is invariant under azimuthal reflections $\phi \to -\phi$ is given by

$$g_{\mu\nu} = \begin{pmatrix} -\frac{V}{r}e^{2\beta} + r^2U^2e^{2\gamma} & -e^{2\beta} & -r^2Ue^{2\gamma} & 0 \\ -e^{2\beta} & 0 & 0 & 0 \\ -r^2Ue^{2\gamma} & 0 & r^2e^{2\gamma} & 0 \\ 0 & 0 & 0 & r^2e^{-2\gamma}\sin^2\theta \end{pmatrix}.$$

Here, γ, β, U and V are functions of (u, r, θ). The Einstein equations impose the following relations on these functions,

$$\partial_r\beta = \frac{1}{2}r(\partial_r\gamma)^2\,, \tag{†}$$

$$\partial_r\left[r^4e^{2(\gamma-\beta)}\partial_rU\right] = 2r^2\left[\partial_r\partial_\theta\beta - \partial_r\partial_\theta\gamma + 2(\partial_r\gamma)\partial_\theta\gamma - 2\cot\theta\,\partial_r\gamma - \frac{2}{r}\partial_\theta\beta\right]. \tag{‡}$$

For an asymptotically flat spacetime with no incoming gravitational radiation, the behaviour of the function γ in the limit $r \to \infty$ is given in terms of the news function c by

$$\gamma(u, r, \theta) = \frac{c(u, \theta)}{r} + \mathcal{O}(r^{-2})\,. \tag{*}$$

(a) By inserting Eq. (*) into Eqs. (†) and (‡), determine the asymptotic behaviour of β and U as $r \to \infty$ to leading order in $1/r$. In both cases, state the leading-order coefficient explicitly in terms of c, its derivatives and functions of integration you may encounter.

(b) Show that $\quad \partial_uc = -\frac{1}{2}\lim_{r\to\infty}\left[r^{-1}(\sin\theta)^{-2}\partial_ug_{\phi\phi}\right]$.

(c) The outcome of two black holes colliding near the speed of light can be described in coordinates $\tilde{x}^{\tilde{\alpha}} = (\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y})$ by the metric

$$\tilde{g}_{\tilde{\alpha}\tilde{\beta}} \begin{pmatrix} 0 & -\kappa^2 & 0 & 0 \\ -\kappa^2 & \kappa\lambda A & \kappa\lambda\frac{\tilde{x}}{\tilde{\rho}}B & \kappa\lambda\frac{\tilde{y}}{\tilde{\rho}}B \\ 0 & \kappa\lambda\frac{\tilde{x}}{\tilde{\rho}}B & \kappa^2 + \kappa\lambda\frac{\tilde{y}^2-\tilde{x}^2}{\tilde{\rho}^2}E & -2\kappa\lambda\frac{\tilde{x}\tilde{y}}{\tilde{\rho}^2}E \\ 0 & \kappa\lambda\frac{\tilde{y}}{\tilde{\rho}}B & -2\kappa\lambda\frac{\tilde{x}\tilde{y}}{\tilde{\rho}^2}E & \kappa^2 + \kappa\lambda\frac{\tilde{x}^2-\tilde{y}^2}{\tilde{\rho}^2}E \end{pmatrix}. \tag{§}$$

Here $\tilde{\rho} = \sqrt{\tilde{x}^2 + \tilde{y}^2}$, κ and λ are positive constants, and A, B and E are functions of $(\tilde{u}, \tilde{\rho})$. The coordinates $\tilde{x}^{\tilde{\alpha}}$ are related to Bondi coordinates (u, r, θ, ϕ) by

$$\tilde{u} = \frac{1}{\sqrt{2}\kappa}(u+r-r\cos\theta)\,, \quad \tilde{v} = \frac{1}{\sqrt{2}\kappa}(u+r+r\cos\theta)\,, \quad \tilde{x} = \frac{r}{\kappa}\sin\theta\,\cos\phi\,, \quad \tilde{y} = \frac{r}{\kappa}\sin\theta\,\sin\phi\,.$$

Show that the Bondi news function of the metric (§) is given by

$$\partial_uc = p\lim_{r\to\infty}(r^q\partial_uE)\,,$$

where p and q are constants you should determine.
[*Hint: You do not need to compute all components of the metric* (§) *in Bondi coordinates.*]

3 Consider a four-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ whose matter content is described by a complex-valued scalar field $\varphi = \varphi_{\rm R} + i\varphi_{\rm I}$ with energy momentum tensor

$$T_{\alpha\beta} = \frac{1}{2}\left\{(\nabla_\alpha\bar{\varphi})(\nabla_\beta\varphi) + (\nabla_\beta\bar{\varphi})(\nabla_\alpha\varphi) - g_{\alpha\beta}\left[g^{\mu\nu}(\nabla_\mu\bar{\varphi})(\nabla_\nu\varphi) + V(|\varphi|^2)\right]\right\}. \qquad (1)$$

Here an overbar denotes the complex conjugate and the potential function V depends only on the squared amplitude $|\varphi|^2 = \bar{\varphi}\varphi$ of the scalar field. Inserting Eq. (1) into the matter equation $\nabla_\mu T^{\mu\alpha} = 0$ yields a wave equation for the scalar field,

$$\nabla^\mu\nabla_\mu\varphi = V'\varphi\,, \qquad \text{where} \qquad V' := \frac{\mathrm{d}V}{\mathrm{d}|\varphi|^2}\,.$$

The Noether current J^μ associated with the scalar field is

$$J^\mu = \frac{i}{2}g^{\mu\nu}(\bar{\varphi}\partial_\nu\varphi - \varphi\partial_\nu\bar{\varphi})\,.$$

In the following, you may assume that the spacetime is globally hyperbolic and admits a 3+1 foliation with future pointing timelike unit normal n^μ, acceleration vector $a_\mu = n^\rho\nabla_\rho n_\mu$ and extrinsic curvature $K_{\mu\nu} := -\perp^\rho{}_\mu\nabla_\rho n_\nu$ where $\perp^\rho{}_\mu = \delta^\rho{}_\mu + n^\rho n_\mu$. In coordinates $x^\mu = (t, x^i)$ adapted to the 3+1 split, the spacetime metric $g_{\mu\nu}$ is related to the lapse α, shift vector β^i and spatial metric γ_{ij} by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\alpha^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i\mathrm{d}t)(\mathrm{d}x^j + \beta^j\mathrm{d}t)\,.$$

Here Latin indices range from 1 to 3 and Greek indices from 0 to 3.

(a) Show that $\nabla_\mu n_\nu = -K_{\mu\nu} - n_\mu a_\nu$.

(b) Show that $a_\mu = \frac{D_\mu\alpha}{\alpha}$, where D_μ is the spatial covariant derivative [*Hint: The components of the unit normal vector are* $n_\mu = -\alpha\nabla_\mu t$.]

(c) Let $\Pi = \Pi_{\rm R} + i\Pi_{\rm I} := -\frac{1}{\alpha}n^\mu\partial_\mu\varphi$ denote the momentum of the scalar field and $Q := -n_\mu J^\mu$ the Noether charge density. Express Q in terms of the real and imaginary parts of φ and Π, and show that Q is real valued.

(d) The spatial Noether current is defined by $Q^\alpha := (\delta^\alpha{}_\mu + n^\alpha n_\mu)J^\mu$. Show that $J^\alpha = Qn^\alpha + Q^\alpha$.

(e) Show that the Noether current is conserved, i.e. $\nabla_\mu J^\mu = 0$.

(f) Show that the conservation law $\nabla_\mu J^\mu = 0$ can be written as

$$n^\mu\nabla_\mu Q = c_1 KQ + c_2 D_\mu Q^\mu + c_3 Q_\nu a^\nu\,, \qquad (\dagger)$$

where c_1, c_2 and c_3 are constants you should determine, and $K = K_\mu{}^\mu$.

(g) Using adapted coordinates (t, x^i) and $n^\mu = (1, -\beta^i)/\alpha$, rewrite Eq. (†) in the form $\partial_t Q = \ldots$ where you should determine the right-hand side in terms of α, β^i, K, Q, Q^i and their derivatives.

END OF PAPER