UNIVERSITY OF CAMBRIDGE

MAMA/355, NST3AS/355, MAAS/355

MAT3

# MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 9:00am to 12:00pm

## PAPER 355

## BIOLOGICAL PHYSICS

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** An elastic filament of constant bending modulus $A$, radius $a$ and length $2L$ lies in the $xy$-plane and is immersed in a fluid of viscosity $\mu$. It is held at its midpoint and oriented along the $y$ axis, while its two ends are free. A constant, uniform fluid flow $\mathbf{u} = U\hat{\mathbf{x}}$, with $U > 0$, is then imposed, bending the filament into $x > 0$. Assume that viscous forces on the filament are given by resistive force theory in the limit $L/a \gg 1$, with parallel and perpendicular drag coefficients $\zeta_{\|}$ and $\zeta_{\perp} = 2\zeta_{\|}$.

a) For suitably small $U$ the filament is weakly deformed from the its straight configuration. Exploiting the symmetry of the problem about the midpoint, let $s \in [0, L]$ be arclength from the midpoint in $y > 0$ of the filament. Adopting a Monge representation with deformation $h(s)$, find the shape to leading order in $U$, expressed as the scaled deformation $H(\xi) = h(s)/L$ from the $y$ axis in terms of the scaled arclength $\xi = s/L$, thereby showing that the Sperm number

$$Sp = \frac{\zeta_{\perp} L^3 U}{A}$$

is the relevant dimensionless parameter of the problem. Estimate the value of $Sp$ beyond which this linearised analysis is expected to break down.

b) The general formulation of this problem involves the relation between the cross-sectionally averaged internal force $\mathbf{F}$ in the filament and the hydrodynamic force $\mathbf{f}$

$$\frac{d\mathbf{F}}{ds} + \mathbf{f} = 0; \quad \text{with } \mathbf{f} = \zeta_{\perp}\left(\hat{\mathbf{n}}\hat{\mathbf{n}} + \frac{1}{2}\hat{\mathbf{t}}\hat{\mathbf{t}}\right) \cdot \mathbf{u}, \tag{1}$$

where $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are the unit tangent and normal vectors. Writing $\mathbf{F} = N\hat{\mathbf{n}} + T\hat{\mathbf{t}}$, where $T$ is the tension and $N = A\kappa_s$, where the curvature $\kappa = \theta_s$, with $\theta(s)$ the tangent angle, find dimensionless versions of the coupled equations for $T$, $N$ and $\kappa$ that govern the shape.

c) For $Sp \gg 1$, the filament is bent strongly by the flow, and far from $s = 0$ the filament will be nearly straight. Find the tension associated with this "outer" solution. Show that there is an inner regime whose arclength scale is $\mathcal{O}(Sp^{-1/2})$ and find the inner solution for the curvature. (You need not find a constant of integration in the solution.)

**2** A population of chemotactic cells with number density $n(\mathbf{r}, t)$ responds to a chemoattractant of concentration $c(\mathbf{r}, t)$ according to the coupled PDEs

$$n_t = D_n \nabla^2 n - \boldsymbol{\nabla} \cdot (n \chi \boldsymbol{\nabla} c) + \gamma - \delta n, \tag{1}$$

$$c_t = D_c \nabla^2 c + \alpha n - \beta c, \tag{2}$$

where the constants $\alpha, \beta, \gamma, \delta, \chi$ and the diffusion constants $D_n$ and $D_c$ are all positive.

a) Find the fixed point values $n^*$ and $c^*$ of the homogeneous system and show that it is linearly stable against homogeneous perturbations.

b) Working now in one spatial dimension, let $n = n^* + u(x, t)$ and $c = c^* + v(x, t)$, and find the linearised dynamics of $u$ and $v$ near the fixed point. Show that the condition for pattern forming instability is

$$\chi > \frac{\delta}{\alpha\gamma} \left( \sqrt{\delta D_c} + \sqrt{\beta D_n} \right)^2 ,$$

and find the critical wavevector $k_c$ at the onset of the instability.

c) Explain the mechanism of the instability.

**3** An elastic filament of constant bending modulus $A$ and length $L$ lies in the $xy$-plane with its left end pinned at $(x, y) = (0, 0)$ and its right end at $(L, 0)$. It is then deformed by a constant force $F$ acting on its right-hand end that compressively buckles the filament while keeping the ends on the $x$-axis. The filament is thus free to pivot at its two ends.

a) Write down the energy $E[\theta]$ of this setup accounting for the bending elasticity of the filament and the work of compression by the force $F$, expressed as a functional of the tangent angle $\theta(s)$. Find the boundary conditions on $\theta(s)$ at the two ends and show that an appropriate Fourier representation of $\theta$ is $\theta(s) = \sum_{n=1}^{\infty} a_n \cos(n\pi s/L)$.

b) Find the critical force $F_c$ for the first Euler buckling mode of the filament. Approximating the shape by the lowest-order mode, calculate the energy $\tilde{E} = E - FL$ to quartic order in $a_1$ and by finding $a_1(F)$ show that there is a pitchfork bifurcation as a function of the parameter $\epsilon = (F - F_c)/F_c$. Find the compression $\mathcal{C}(F) = 1 - L_{\|}/L$ in the neighbourhood of $F_c$, where $L_{\|}$ is the projected length of the filament along the $x$-axis.

c) Now consider the filament at temperature $T$. Expand $\tilde{E}[\theta]$ to quadratic order around the straight configuration, and show that $\langle a_n \rangle = 0$ and

$$\langle a_n^2 \rangle = \frac{2k_B T L}{A\pi^2 n^2 - FL^2}, \tag{1}$$

where $k_B$ is Boltzmann's constant. Explain the physical significance of the divergence of $\langle a_1^2 \rangle$ at $F_c$. Find $\mathcal{C}(0)$ in terms of the persistence length $L_p$, and interpret the result. [Hint: $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$].

d) At a positive temperature $T$, Gaussian fluctuations generate thermal rounding of the $T = 0$ pitchfork bifurcation found in (b). An estimate of this effect can be obtained by using a reference state that is an effective quadratic theory with energy function $U_0 = L g_1 a_1^2$, where $g_1$ is a variational parameter, and treating the energetic terms as a perturbation. To leading order, the free energy is

$$\mathcal{F}(g_1) = \mathcal{F}_0(g_1) + \langle \tilde{E} \rangle_0, \tag{2}$$

where $\mathcal{F}_0(g_1) = -k_B T \ln Z_0$, with $Z_0$ the partition function associated with the energy $U_0$, and the angular brackets in (2) denote an average with respect to the reference system. Here, $\tilde{E}$ is to be taken up to quartic order in $\theta$, assuming a single-mode approximation. Find $\mathcal{F}(g_1)$, minimize it with respect to $g_1$, and thus obtain an optimum estimate of $\mathcal{C}(F)$. Show that it exhibits thermal rounding of the $T = 0$ transition found in (b), controlled by the ratio $L/L_p$.

## END OF PAPER