MAMA/354, NST3AS/354, MAAS/354

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 16 June 2026 2:00pm to 4:00pm

PAPER 354

GAUGE/GRAVITY DUALITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define a conformal transformation on flat d–dimensional Euclidean space. Derive the condition that an infinitesimal coordinate transformation

$$\tilde{x}^\mu = x^\mu + \delta\, \varepsilon^\mu(x) + \mathcal{O}(\delta^2)$$

must satisfy in order to be conformal.

(b) Consider a scalar primary operator $\mathcal{O}(x)$ of scaling dimension Δ in a d-dimensional Euclidean CFT.

(i) Using conformal invariance, show that the two-point function of $\mathcal{O}$ is fixed (up to an overall normalisation).

(ii) Derive the form of the 3-point function of three scalar primary operators $\mathcal{O}_i$ with scaling dimensions Δ_i.

(iii) Assume that the CFT is holographic, and that the scalar operator(s) $\mathcal{O}$ are single-trace. Sketch Witten diagrams contributing to the two- and three-point functions described above.

(c) Let $\mathcal{O}$ be a scalar primary of dimension Δ in a d-dimensional CFT. Consider the operator product expansion (OPE)

$$\mathcal{O}(x)\, \mathcal{O}(0) \sim \frac{C}{|x|^{2\Delta}} + \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}}}{|x|^{\Delta}}\, \mathcal{O}(0) + \cdots$$

Using your results from part (b) for the two- and three-point functions:

(i) Show that inserting this OPE into the three-point function $\langle \mathcal{O}(x)\mathcal{O}(0)\mathcal{O}(y)\rangle$ correctly reproduces the form fixed by conformal invariance when $|x| \ll |y|$.

(ii) Hence determine the proportionality relation between $C_{\mathcal{O}\mathcal{O}\mathcal{O}}$ and the normalisation constants appearing in your two- and three-point functions.

2

Consider the four–dimensional Einstein–Maxwell action with a negative cosmological constant

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(R - 2\Lambda - F_{ab}F^{ab}\right), \tag{$\star$}$$

where $F = \mathrm{d}A$ and G is Newton's constant.

(a) Derive the equations of motion obtained from the action ($\star$). You may ignore all boundary terms in this calculation.

(b) A planar charged AdS black hole solution to these equations is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}x^2 + \mathrm{d}y^2) \qquad A = \left(\frac{Q}{r_+} - \frac{Q}{r}\right)\mathrm{d}t\,, \tag{$\dagger$}$$

with

$$f(r) = \frac{r^2}{L^2} - \frac{2M}{r} + \frac{Q^2}{r^2}\,,$$

where Q and $M > 0$ are the charge and mass parameters of the black hole, L is the AdS length scale, and the event horizon is given by the largest real zero of $f(r)$, $r = r_+$.

Perform the coordinate transformations

$$r = \frac{r_+}{z}, \quad t = \frac{L^2}{r_+}\tau, \quad x = \frac{L}{r_+}X, \quad y = \frac{L}{r_+}Y\,,$$

and define $Q \equiv r_+^2\mu/L$. Show that in the new coordinates the solution ($\dagger$) depends nontrivially only on the dimensionless parameter μ.

(c) Rewrite the metric and gauge field ($\dagger$) in Fefferman–Graham coordinates (Z, τ, X, Y) and expand them up to $\mathcal{O}(Z^2)$. Using the holographic dictionary, interpret the physical meaning of the parameter μ in the dual CFT.

(d) Consider now the coordinate transformation

$$\tau = T\cosh\beta + W\sinh\beta \quad \text{and} \quad X = T\sinh\beta + W\cosh\beta\,.$$

Explain the physical meaning of this transformation and compute the CFT current induced by it.

(e) Determine the D.C. conductivity associated with this system. Is your result physically reasonable? Justify your answer.

END OF PAPER