UNIVERSITY OF CAMBRIDGE

MAMA/353, NST3AS/353, MAAS/353

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 11 June 2026 9:00am to 12:00pm

PAPER 353

NOISY MECHANICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the multivariate Orstein-Uhlenbeck (OU) process $\dot{x}_i = -A_{ij}x_j + b_{ij}\Lambda_j(t)$, with Λ_i uncorrelated unit white noises. Here $\mathbf{A}$ and $\mathbf{b}$ are constant matrices, and $\mathbf{A}$ has (right) eigenvalues with only positive real parts. For this process it may be shown that in steady state $\langle x_i(t)\rangle = 0$, and that $\langle x_i(t)x_j(s)\rangle \equiv G_{ij}(t,s)$ obeys

$$\mathbf{G}(t,s) = \exp[-(t-s)\mathbf{A}].\boldsymbol{\sigma} \ \text{ for } t > s \quad ; \quad \mathbf{G}(t,s) = \boldsymbol{\sigma}.\exp[-(s-t)\mathbf{A}^T] \ \text{ for } s > t$$

where $\boldsymbol{\sigma}$ obeys $\mathbf{A}.\boldsymbol{\sigma} + \boldsymbol{\sigma}.\mathbf{A}^T = \mathbf{b}.\mathbf{b}^T$. (You are not asked to show the above results here.)

(a) (i) Noting that $\mathbf{G}(t,s)$ can be written $\mathbf{G}(t-s)$, establish the result $\mathbf{G}(\tau) = \mathbf{G}(-\tau)^T$.

(ii) For $\mathbf{s}(\omega) \equiv \int_{-\infty}^{\infty} e^{-i\omega\tau}\mathbf{G}(\tau)\,d\tau$, establish that $(\mathbf{A}+i\omega\mathbf{I}).\mathbf{s}(\omega).(\mathbf{A}^T - i\omega\mathbf{I}) = \mathbf{b}.\mathbf{b}^T$.

(b) A damped thermal simple harmonic oscillator (of mass m, damping constant γ and spring constant λ) in one spatial dimension obeys

$$\begin{aligned}\dot{x} &= \frac{p}{m}\\ \dot{p} &= -\frac{\gamma}{m}p - \lambda x + \sqrt{2k_BT\gamma}\,\Lambda_2(t).\end{aligned}$$

(i) Write down the matrices $\mathbf{A}$ and $\mathbf{b}.\mathbf{b}^T$ for the two-variable OU process obeyed by $\mathbf{x} = (x_1, x_2) \equiv (x, p)$. Find the equal time correlation matrix $\boldsymbol{\sigma}$ and show that it meets the requirements of the Boltzmann distribution, $P(\mathbf{x}) \propto \exp[-H(x,p)/k_BT]$, including the requirement that $\sigma_{12} = 0$.

(ii) Calculate the power spectrum matrix $\mathbf{s}(\omega)$ as defined in part (a), and show that its xx component equates to

$$s_{xx}(\omega) = \frac{2k_BT\gamma}{m^2(\omega^2 - \lambda/m)^2 + \gamma^2\omega^2}\,.$$

(c) The trajectory $x(t)$ of a certain particle obeys

$$\begin{aligned}\dot{x} &= \mu(-\lambda x + q) &\quad (1)\\ \dot{q} &= -\nu q + \sqrt{\alpha\nu}\,\Lambda_2(t)\,. &\quad (2)\end{aligned}$$

Here μ, λ, ν and α are positive constants.

(i) Show from (2) that in steady state $\langle q(t)q(t+\tau)\rangle = \frac{\alpha}{2}e^{-\nu|\tau|}$, and hence that (1) can be interpreted as describing an overdamped harmonic particle with mobility μ subject to a zero-mean random force q having nonzero autocorrelation time (*i.e.*, non-white noise).

(ii) Find $\boldsymbol{\sigma}$ for the system (1,2).

(iii) Show that white noise appropriate to an overdamped thermal particle is recovered in the limit $\nu \to \infty$, taken such that $\frac{\alpha}{\nu}$ = constant $= 2k_BT/\mu$, and that the correct equal-time variance of x is recovered in this limit.

2 (a) Consider a system with $\mathbb{D} \geqslant 2$ coordinates $\mathbf{x}(t) \in \mathbf{X}$, obeying a Fokker-Planck equation in the form

$$\dot{P}(x_i, t) = -\nabla . \boldsymbol{\mathcal{J}} \equiv -\partial_i \mathcal{J}_i = -\partial_i \left(a_i P - \frac{1}{2} \kappa_{ij} \partial_j P \right) ,$$

where a_i and $\kappa_{ij} = \kappa_{ji}$ are arbitrary functions of the coordinates $\mathbf{x}$.

(i) Show that if the following potential conditions hold

$$\frac{\partial}{\partial x_l} (\kappa_{ji}^{-1} a_i) = \frac{\partial}{\partial x_j} (\kappa_{li}^{-1} a_i) ,$$

then the probability current $\boldsymbol{\mathcal{J}}$ can be written for some choice of tensor $\mathbf{D}(\mathbf{x})$ and scalar $V(\mathbf{x})$ (which should both be identified) as

$$\boldsymbol{\mathcal{J}}(\mathbf{x}) = -P(\mathbf{x}, t) \mathbf{D}(\mathbf{x}) . \nabla (V(\mathbf{x}) + \ln P(\mathbf{x}, t)) .$$

You may assume without proof that $\partial_{x_l} \mathcal{C}_j = \partial_{x_j} \mathcal{C}_l \Rightarrow \mathcal{C}_j = \partial \Psi / \partial x_j$ for some $\Psi(\mathbf{x})$.

(ii) Show that for positive definite $\mathbf{D}$ the functional

$$\mathcal{F} = \int_{\mathbf{X}} [V(\mathbf{x}) P(\mathbf{x}, t) + P(\mathbf{x}, t) \ln P(\mathbf{x}, t)] \, d\mathbf{x}$$

obeys $\dot{\mathcal{F}} < 0$ unless $\boldsymbol{\mathcal{J}} = \mathbf{0}$. You may assume that $\boldsymbol{\mathcal{J}}$ vanishes on the boundary of $\mathbf{X}$ and that $P(\mathbf{x}, t)$ is normalized to unity.

(iii) Show that $\mathcal{F}$ is minimized by $P = P_B \equiv e^{-V(\mathbf{x})} / \int e^{-V(\mathbf{x}')} \, d\mathbf{x}'$, and hence that if $\mathcal{F}$ is bounded below, P_B is the steady state probability density and is unique.

(b) A model of interacting bacteria describes the population densities $x_i \geqslant 0$ of two species $i = 1, 2$ with the equations of motion (in suitable units)

$$\begin{aligned} \dot{x}_1 &= r_1 x_1 (1 - c_1 x_2^2) - d_1 x_1^2 + b_1 \Lambda_1(t) \\ \dot{x}_2 &= r_2 x_2 (1 - c_2 x_1^2) - d_2 x_2^2 + b_2 \Lambda_2(t) . \end{aligned}$$

Here r_i, c_i, d_i, b_i are positive constants and $\Lambda_i(t)$ are independent unit white Gaussian noises. Thus each species has its own baseline cell-division rate r_i and crowding-induced death rate d_i. The coupling terms with coefficients c_i model an effect whereby when an individual of one species dies and decomposes, a chemical is released that is mildly toxic to the other species only, inhibiting its cell-division. (You may assume that, for all densities of interest, the resulting division rates $r_1(1 - c_1 x_2^2)$ and $r_2(1 - c_2 x_1^2)$ remain positive.)

(i) Write out the Fokker Planck equation for the probability density $P(x_1, x_2, t)$, and show that the potential condition is

$$r_1 c_1 b_2^2 = r_2 c_2 b_1^2 . \tag{1}$$

(ii) For the case where (1) is satisfied, write an expression for the steady state probability density $P_{\mathrm{ss}}(x_1, x_2)$. You may assume that $\boldsymbol{\mathcal{J}} = 0$ on the lines $x_1 = 0$ and $x_2 = 0$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Now consider a new model where the decomposition product is not a toxin but a nutrient, so that the couplings obey $c_i < 0$. Taking (**??**) still to hold, and $\mathcal{J}$ still to vanish on $x_1 = 0$ and $x_2 = 0$, say what you can about the long-time dynamics predicted by the new model. (Detailed calculation is not expected.)

3 Consider a model that couples a conserved order parameter $\phi(\mathbf{r},t)$ and a non-conserved one $\psi(\mathbf{r},t)$ as follows (in units with $k_BT = 1$):

$$\dot{\phi} = -\nabla.\mathbf{J} \quad ; \quad \mathbf{J} = -M\nabla\frac{\delta F}{\delta\phi} + \sqrt{2M}\boldsymbol{\Lambda}(\mathbf{r},t)\,,$$
$$\dot{\psi} = -\Gamma\frac{\delta F}{\delta\psi} + \sqrt{2\Gamma}\Lambda(\mathbf{r},t)$$

with $\boldsymbol{\Lambda}$ and Λ independent (vectorial and scalar) unit white spatiotemporal noises. Here M, Γ are constants and the free energy functional is

$$F[\phi,\psi] = \int \left(\frac{a}{2}\phi^2 + \frac{\bar{a}}{2}\psi^2 + c\phi\psi + \frac{\kappa}{2}(\nabla\phi)^2 + \frac{\bar{\kappa}}{2}(\nabla\psi)^2\right) d\mathbf{r}\,, \tag{1}$$

with $a, \bar{a}, c, \kappa, \bar{\kappa}$ further constants.

(i) Defining the Fourier mode amplitudes as usual via $\phi_{\mathbf{q}} = \int e^{-i\mathbf{q}.\mathbf{r}}\phi(\mathbf{r})d\mathbf{r}$ and $\phi(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{q}} e^{i\mathbf{q}.\mathbf{r}}\phi_{\mathbf{q}}$, *etc.*, give the equations of motion in terms of $\phi_{\mathbf{q}}, \psi_{\mathbf{q}}, \boldsymbol{\Lambda}_{\mathbf{q}}, \Lambda_{\mathbf{q}}$. (You are not asked for the explicit statistics of $\boldsymbol{\Lambda}_{\mathbf{q}}$ or $\Lambda_{\mathbf{q}}$.)

(ii) Find a $\mathbf{q}$-dependent two-by-two matrix acting on the vector $(\phi_{\mathbf{q}}, \psi_{\mathbf{q}})$, whose right eigenvectors are the hydrodynamic modes, and give a quadratic equation obeyed by its two eigenvalues.

(iii) Now specialize to the case where $\Gamma = M = 1$ and $a = \bar{a}, \kappa = \bar{\kappa}$. Find the eigenvalues perturbatively in small coupling constant c. Show that to order q^2 these are

$$\lambda_B = q^2(a - c^2/a) \quad ; \quad \lambda_A = a + (\kappa + c^2/a)q^2\,.$$

(iv) Again working to order q^2, show that the resulting (non-normalized) eigenvectors are respectively $\mathbf{e}_B = (1,\beta)$ and $\mathbf{e}_A = (\alpha, 1)$, with $\beta = -c/a$ and $\alpha = cq^2/a$.

(v) Decomposing a specific initial state with $\phi_{\mathbf{q}}(0) \neq 0$ and $\psi_{\mathbf{q}}(0) = 0$ as a superposition of the hydrodynamic modes, show that in the small q regime at subsequent times t, the mean mode amplitude $\langle\phi_{\mathbf{q}}(t)\rangle$ obeys $\langle\phi_{\mathbf{q}}(t)\rangle = \phi_{\mathbf{q}}(0)\mathcal{G}_\phi(t)$, where

$$\mathcal{G}_\phi(t) = f_1(q)e^{-\lambda_B t} + f_2(q)e^{-\lambda_A t}\,.$$

Give explicitly the functions f_1, f_2 to order q^2 and explain carefully why the conservation law on ϕ requires f_2 to vanish as $q \to 0$.

(vi) Consider now, instead of small $\mathbf{q}$, a wavevector with $|\mathbf{q}| = 1$. Again specialising to the case where $\Gamma = M = 1$ and $a = \bar{a}, \kappa = \bar{\kappa}$, find the hydrodynamic mode eigenvalues and eigenvectors. Hence calculate $\mathcal{G}_\phi(t)$ for the $|\mathbf{q}| = 1$ case.

(vii) For the *same* initial state as considered before, *i.e.*, $\phi_{\mathbf{q}}(0) \neq 0$ and $\psi_{\mathbf{q}}(0) = 0$, write down for $|\mathbf{q}| = 1$ the time-dependent average $\langle\psi_{\mathbf{q}}(t)\rangle$.

(viii) With reference to the form of F in (1), or otherwise, find (for $a = \bar{a}, \kappa = \bar{\kappa}$, and $|\mathbf{q}| = 1$) the equal-time cross-correlator $\langle\psi^*_{\mathbf{q}}(t)\phi_{\mathbf{q}}(t)\rangle$ in the steady state ($t \to \infty$).

4 (a) Consider a process in which the probability of exactly one event occurring in a time interval dt is μdt, where μ is a positive constant. The probability of exactly N_t events occurring in time t is given by the Poisson distribution

$$P(N_t = n) = \frac{(\mu t)^n}{n!} e^{-\mu t}.$$

(i) Derive the moment generating function $G(\lambda) = \mathbb{E}(e^{\lambda N_t})$ for this process and hence obtain the mean and the variance of N_t.

(ii) Consider the number of events per unit time $X_t = N_t/t$ over a duration t. Derive the rate function $I(x)$ for this quantity using Cramer's theorem,

$$I(x) = \sup_{\lambda} \left[\lambda x - \frac{1}{t} \ln G(\lambda) \right].$$

(iii) Aircraft arrivals at an airport follow a Poisson process with rate μ_1 arrivals per hour. The airport can handle at most μ_2 arrivals per hour without facing congestion. Use the rate function to obtain an exponentially equivalent estimate of the probability of congestion over time t. Suppose the handling capacity is increased to a rate $\mu_3 > \mu_2$. Determine a condition on μ_3 such that the probability of congestion over a time interval of length t is reduced by a factor of at least 10^6 relative to the capacity μ_2. Express your answer as an inequality involving the rate function and simplify it as far as possible.

(b) Let W_t, $t \geqslant 0$, be the Wiener process.

(i) State the probability distribution of $\Delta W = W_{t+h} - W_t$.

(ii) Show that for fixed t, $\mathbb{E}(W_{t+h} - W_t)^2 = h$ and argue briefly why this suggests that W_t is not differentiable as a function of t.

(c) Consider the n-dimensional Itô process

$$dx = a(x)\, dt + \sqrt{\epsilon}\, \sigma(x)\, dW,$$

where $x \in \mathbb{R}^n$, $a(x)$ is the n-dimensional drift vector field and $\sigma(x)$ is the $n \times n$ diffusivity matrix. The diffusion matrix $2D = \sigma\sigma^T$ is full rank and $\epsilon \ll 1$. The Freidlin–Wentzell rate functional for x_t to remain in the neighbourhood of the smooth path $\varphi(t)$ is given by

$$I[\varphi] = \frac{1}{2} \int_0^T (\dot{\varphi} - a)^T (\sigma\sigma^T)^{-1} (\dot{\varphi} - a)\, dt.$$

(i) Show that for paths of infinite duration, the rate functional can be expressed as

$$I[\varphi] = \int \|a\|\, ds - a \cdot d\varphi,$$

where you may assume that the Hamiltonian corresponding to $I[\varphi]$ vanishes for paths of infinite duration and for two vectors $\alpha, \beta \in \mathbb{R}^n$,

$$\alpha \cdot \beta = (2D^{-1})_{ij} \alpha_i \beta_j, \qquad \|\alpha\| = (\alpha \cdot \alpha)^{1/2}.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

(ii) Consider an overdamped particle diffusing in a two-dimensional potential $V(x_1, x_2)$ satisfying detailed balance with constant diffusion matrix $D_{ij} = D\delta_{ij}$. Show that the escape path from a minimum of V is always normal to its equipotentials.

END OF PAPER