UNIVERSITY OF CAMBRIDGE

MAMA/352, NST3AS/352, MAAS/352

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 4 June 2026 9:00am to 12:00pm

PAPER 352

NON-NEWTONIAN FLUID MECHANICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Derive the sliding law experienced by a large-scale flow moving over a thin layer of Bingham fluid of depth h with yield strength τ_c and constant viscosity μ after yield, below which lies a stationary no-slip wall.

(b) An *unconfined ice stream margin* consists of a layer of power law fluid flowing down a slope of angle $\theta \ll 1$, over a base that sharply transitions between materials. Let the fluid have power index $n \leqslant 1$, consistency K, density ρ and uniform depth H, with n, K, ρ, H, and θ all constant. Take (x, y, z) to be standard Cartesian coordinates where x is the down-slope coordinate, y the cross-slope coordinate, and z an upwards coordinate normal to the base. Let the velocity of the power-law fluid be $\mathbf{u} = (u(y, z), 0, 0)$.

Suppose that the base consists of a thin layer of Bingham fluid of constant depth $h \ll H$ over a stationary no-slip wall. The Bingham fluid has a spatially dependent yield stress, such that the shear stress τ and shear rate $\dot{\gamma}$ are related by

$$\tau = \tau_c + \mu\dot{\gamma} \quad \text{when} \quad |\tau| \geqslant \tau_c; \qquad \tau_c = \begin{cases} \tau_1, & y < 0, \\ \tau_2, & y > 0, \end{cases} \tag{1}$$

with $\dot{\gamma} = 0$ when $|\tau| < \tau_c$, where $\tau_1 < \tau_2 < \rho g H \sin\theta$ and μ is a constant.

(i) Write down the components of the force balance and the boundary conditions for the flow of the unconfined ice stream.

(ii) Suppose that horizontal velocity gradients dominate the shear rate within the power-law fluid, so $u = U(y) + \epsilon\tilde{u}(y, z)$ with $\epsilon \ll 1$. Integrate the force balance to derive an ODE satisfied by $U(y)$. What is the far-field flow speed of the ice stream as $y \to \pm\infty$? Sketch the approximate shape of $U(y)$.

(iii) Using dU/dy as an integrating factor, or otherwise, find dU/dy as a function of U. Hence find $U(y)$ in the case $n = 1$. For $n < 1$, find $U(y)$ in $y > 0$ up to an integration constant which you need not determine. Sketch $U(y)$ in these cases and comment on what this might reveal about the rheology of ice in a geophysical context.

(iv) In what limits, and in which regions of the flow, is the assumption of dominant horizontal shear valid?

(v) Comment briefly on the case $n > 1$.

2 The stress $\boldsymbol{\tau}$ and rate of strain $\dot{\boldsymbol{\gamma}}$ in a general class of viscoelastic fluids are related by

$$\boldsymbol{\tau} = \eta_s \dot{\boldsymbol{\gamma}} + \frac{\eta_p}{\lambda}\left(F\boldsymbol{C} - \boldsymbol{I}\right); \qquad \overset{\triangledown}{\boldsymbol{C}} = -\frac{1}{\lambda}\left(F\boldsymbol{C} - \boldsymbol{I}\right),$$

where $\boldsymbol{C}$ is the conformation tensor, $\boldsymbol{I}$ the identity tensor, λ, η_s and η_p are the constant relaxation time, solvent viscosity and polymeric viscosity, respectively, $\overset{\triangledown}{\boldsymbol{C}}$ denotes the upper convected derivative, and F is a function that depends on $T \equiv \mathrm{trace}(\boldsymbol{C})$. An Oldroyd-B fluid has $F = 1$, and a FENE-P fluid has $F(T) = (L^2 - 3)/(L^2 - T)$, where $L^2 > 3$ is a dimensionless constant.

(a) Why would replacing the convected derivative in the model above with a material derivative be problematic? Briefly outline some features of the Oldroyd-B model that might make it an inappropriate model for typical polymeric fluids.

(b) Consider the steady response of a FENE-P fluid to a simple shear flow $\boldsymbol{u} = (\dot{\gamma} y, 0, 0)$.

(i) Calculate the components of the stress tensor $\boldsymbol{\tau}$ in terms of F. Which normal-stress differences can the FENE-P model exhibit?

(ii) Show that F satisfies a cubic equation of the form $F^2(F-1) = \beta$, where β depends on the parameters of the problem and the shear rate. Determine the limiting behaviour of F for large and small $\dot{\gamma}$, and hence sketch the effective shear viscosity of the FENE-P material as a function of shear rate, comparing with the equivalent result for an Oldroyd-B fluid.

(c) A FENE-P fluid is subjected to a steady uniaxial extension flow in the z direction with constant imposed extensional strain rate $\dot{\epsilon} > 0$.

(i) Find expressions for all non-zero components of the conformation tensor $\boldsymbol{C}$ in terms of the parameters of the problem and F. Determine the extensional viscosity η_{ext} of the fluid, also in terms of F, and an implicit equation that determines $F(T)$.

(ii) Show that a FENE-P fluid and an Oldroyd-B fluid share the same extensional viscosity for small extension rates and determine the limiting value $\eta_{ext}(\dot{\epsilon} \to 0)$.

(iii) Calculate the behaviour of η_{ext} in the limit $\dot{\epsilon} \gg 1/\lambda$, and hence sketch the extensional viscosity $\eta_{ext}(\dot{\epsilon})$ for both a FENE-P and an Oldroyd-B fluid.

3 Consider a dense suspension of rigid particles in an incompressible fluid of constant viscosity η. The particles and fluid have equal densities. A rheometric test is performed on a sample of this suspension, in which the volume of the sample is fixed and a shear rate $\dot{\gamma}$ is imposed on the solid particles. The associated shear stress τ and solid pressure p_s in steady state are recorded, and no normal-stress differences are detected. Provided the solid volume fraction ϕ is less than some constant maximum value ϕ_m, the data fits

$$\tau = \eta f_1(\phi)\dot{\gamma}; \qquad p_s = \eta f_2(\phi)\,|\dot{\gamma}|\,,$$

where

$$f_1(\phi) = 1 + \hat{\mu}\frac{\phi^2}{(\phi_m - \phi)^2}; \qquad f_2(\phi) = \frac{\phi^2}{(\phi_m - \phi)^2},$$

for constant $\hat{\mu} > 0$. If $\phi = \phi_m$, the system is jammed and there is no motion ($\dot{\gamma} = 0$).

(a) The rheometer is adjusted to instead impose a fixed solid pressure p_s on the sample, as well as the shear rate $\dot{\gamma}$, using a movable porous plate surrounded by the saturating fluid, such that the solid fraction ϕ can evolve. What expressions would you expect to measure for the steady shear stress τ and solid fraction ϕ now? If the rheometer imposes a step change in the shear rate, why would you expect the stress to take some time to evolve to its new steady state, and what additional property of the suspension might this timescale provide information about?

(b) The suspension is now pumped slowly along a horizontal pipe of circular cross-section, radius R and length L, driven by an imposed fluid pressure drop ΔP. You should assume that the solid pressure p_s is uniform throughout the pipe, that inertial forces are negligible, and that the particles do not slip on the pipe walls.

(i) Find the minimum ΔP that is required to move the particles. Assuming ΔP exceeds this value, find the steady velocity profile of the solid particles along the pipe, $w_s(r)$, and the associated solid volume fraction, $\phi(r)$.

(ii) Fluid can seep around the solid particles in the suspension according to Darcy's law, with a differential velocity given by $-k/\eta\,\partial p/\partial z$, where p is the fluid pressure, z is the coordinate along the pipe and $k \ll R^2$ is the permeability of the packing, which you should assume to be constant. What is the total flux Q_T of the suspension?

(iii) For a given ΔP, sketch how the total flux varies with the solid pressure p_s. The particles become motionless once the solid pressure exceeds a certain value p_s^*; find an expression for the flux of particles Q_p when the solid pressure is close to this value. Over what range of $p_s < p_s^*$ does the seeping flow of fluid through the suspension have an appreciable influence on the total flux?

(iv) Suppose now that the total flux Q_T is fixed. Sketch how the particle flux Q_p varies with p_s in this case (you do not need to compute Q_p) and hence show that the same particle flux can be achieved for two different solid pressures. Compare the structure of the flow and the associated pressure drop ΔP in these two cases.

END OF PAPER