MAMA/347, NST3AS/347, MAAS/347

MAT3

MATHEMATICAL TRIPOS Part III

Friday 12 June 2026 2:00pm to 4:00pm

PAPER 347

ASTROPHYSICAL BLACK HOLES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Explain what is meant by the concept of radiative efficiency, and why maximum radiative efficiency depends on the black hole spin.

By contrasting adiabatic Bondi accretion flow with the accretion flow through a Shakura-Sunyaev accretion disc, argue why, in order to achieve high radiative efficiencies, a source of heating more efficient than the adiabatic compression is needed.

Hence, explain which physical process is most likely responsible for providing this source of heating in standard thin discs, and why the α-disc prescription adopted in the Shakura-Sunyaev disc solution provides a viable approach to capture this physical mechanism. Furthermore, explain three limitations of the α-disc prescription.

(b) Consider a differentially rotating disc with angular velocity $\Omega_0 \propto r^{-q}$, where q is constant. Initially, the disc is characterised by a constant density ρ_0, with an isothermal equation of state with sound speed $c_{\rm s,0}$ being constant. The disc is threaded by a uniform magnetic field B_0 along the z-axis. At some radius in the disc r_0 consider small perturbations in density ρ, pressure p, velocity $\vec{v}$ and magnetic field $\vec{B}$. If the disc satisfies the following equations in the shearing box approximation, i.e. with x and y being local Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0\,, \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}(\nabla \times \vec{B}) \times \vec{B} - 2\Omega_0 \hat{z} \times \vec{v} + 2q\Omega_0^2 x\hat{x}\,, \tag{2}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})\,, \tag{3}$$

$$\nabla \cdot \vec{B} = 0\,, \tag{4}$$

write down the governing equation for the unperturbed system and determine the initial velocity field.

Now consider perturbations of the form $B_x^{'} = \bar{B}_x^{'} e^{i(\omega t - kz)}$ and similarly for $B_y^{'}$, $\rho^{'}$, $p^{'}$ and $\vec{v}^{\,'}$, but note that $B_z^{'}$ is constant. Assume that the perturbations depend on z and t only, where ω is the frequency and k is the wave number of a perturbation. Hence, analyse the x and y components of the momentum and induction equation, neglecting the second order perturbation terms, to derive the following dispersion relation:

$$\omega^4 - 2\omega^2(k^2 v_{\rm A}^2 - \Omega_0^2(q-2)) - 2q\Omega_0^2 k^2 v_{\rm A}^2 + k^4 v_{\rm A}^4 = 0\,. \tag{5}$$

Here you may assume that the Alfvén speed is $v_{\rm A}^2 = B_0^2/(4\pi\rho_0)$.

Using (5), or otherwise, show that the condition for instability reads:

$$k^2 v_{\rm A}^2 + \frac{d\,\Omega_0^2}{d\ln r} < 0\,. \tag{6}$$

Now assuming that this disc is geometrically thin and Keplerian, show that the maximum growth rate of the instability happens for $|\omega_{\rm max}| = 3\Omega_0/4$. Comment on whether this instability growth is likely to be large or small.

Furthermore, by imposing that the smallest unstable scale of the instabilities $\lambda_{\rm crit}$ is comparable to $2\,H$, where H is the disc vertical thickness, show that this corresponds to $B_{0,\rm crit}^2 = 12\pi^{-1}\rho_0 c_{\rm s,0}^2$. What is the physical meaning of this result?

2 (a) Consider a system with size R composed of N identical stars with mass m uniformly distributed and subject to the two body relaxation process. Show that the change in velocity v during one encounter between two stars at separation r and characterised by the impact parameter b is given by:

$$\delta v = \frac{2Gm}{b^2}\frac{b}{v}\,, \tag{1}$$

where G is the gravitational constant, and explain the physical meaning of the terms on the right-hand side of the equation.

Now, integrating over all possible impact parameters for all encounters in this system, and assuming that the typical velocity of stars is given by the virial theorem, derive the number of crossings $n_{\rm cross}$ needed such that the system loses memory of the initial velocity.

Using the derived expression for $n_{\rm cross}$, write down the expression detailing how the relaxation time $t_{\rm rel}$, depends on N, v and R.

What is the necessary condition for this system to be collisional?

(b) Consider a black hole of mass $M_{\rm BH}$ residing at the centre of a collisional star cluster. A star of mass M_* and radius R_* approaching sufficiently close to the black hole may be tidally disrupted if the tidal force from the black hole is strong enough. Estimate this tidal disruption radius to show that there is a critical black hole mass:

$$M_{\rm BH,crit} = \frac{c^3}{2} M_*^{-\frac{1}{2}} \left(\frac{R_*}{G}\right)^{\frac{3}{2}}\,, \tag{2}$$

such that the star will be swallowed whole, rather than tidally disrupted. Here c is the speed of light.

Does this critical black hole mass depend on the black hole spin?

Assuming that the masses of intermediate mass black holes are lower than $M_{\rm BH,crit}$, what does this result imply for the growth and detection of intermediate mass black holes residing in dense star clusters?

[Hint: To estimate the tidal disruption radius you may equate the surface gravity of the star and the tidal acceleration exerted by the black hole, $a_{\rm tide} \simeq (2GM_{\rm BH}R_*)/r^3$, where r is the distance between the black hole and the star.]

[QUESTION CONTINUES ON THE NEXT PAGE]

(c) Consider now an advection dominated accretion flow (ADAF) in a steady state, in the gas pressure dominated regime, whereby kinematic shear viscosity can be expressed through the α-prescription. Assume that this disc is governed by the usual set of equations:

$$\frac{d}{dR}(\rho R H u_R) = 0\,, \tag{3}$$

$$u_R \frac{du_R}{dR} = (\Omega^2 - \Omega_{\rm K}^2)R - \frac{1}{\rho}\frac{d}{dR}(\rho c_{\rm s}^2)\,, \tag{4}$$

$$u_R \frac{d(\Omega R^2)}{dR} = \frac{1}{\rho R H}\frac{d}{dR}\left(\frac{\alpha \rho c_{\rm s}^2 R^3 H}{\Omega_{\rm K}}\frac{d\Omega}{dR}\right), \tag{5}$$

$$2\rho H u_R T \frac{ds}{dR} = Q^+ - Q^- = f\,\frac{2\alpha\rho c_{\rm s}^2 R^2 H}{\Omega_{\rm K}}\left(\frac{d\Omega}{dR}\right)^2, \tag{6}$$

where R is the cylindrical radius, ρ is the vertically-averaged density, H is the vertical scale height, u_R is the radial velocity, Ω is the angular velocity, $\Omega_{\rm K}$ is the Keplerian angular velocity, $c_{\rm s}$ is the sound speed, and the gas pressure $p = \rho c_{\rm s}^2$. In the energy equation s is the entropy, T is the temperature, Q^+ is the viscous dissipation term and Q^- is the radiative cooling term, with the parameter f measuring the degree to which the flow is advection-dominated. If $\epsilon = (5/3-\gamma)/(\gamma-1)$, where γ is the adiabatic index, and $\epsilon' = \epsilon/f$ is independent of R, the ADAF equations above admit self-similar solutions of the form:

$$\rho = c_0 R^{-3/2}, \quad u_R = c_1 v_{\rm K}, \quad \Omega = c_2 \Omega_{\rm K}, \quad c_{\rm s}^2 = c_3 v_{\rm K}^2\,, \tag{7}$$

where $v_{\rm K}$ is the Keplerian velocity, and c_0, c_1, c_2 and c_3 are constants. Calculate the constants c_1, c_2 and c_3, and also determine their values in the limit of α^2 being small.

Using these results, explicitly demonstrate that in the limit of very efficient cooling, this solution corresponds to the standard thin disc solution.

In case of a significant advection show quantitatively how u_R, Ω and $c_{\rm s}$ deviate from the standard thin disc solution and explain the physical nature of this flow, also focusing on the limit where $\gamma \to 5/3$.

Give an example of a supermassive black hole whose accretion flow is well described by the ADAF solution, and justify your answer.

3

(a) Explain in detail through which physical processes black hole seeds may originate considering the three main seeding channels: (i) population III star remnant, (ii) runaway collapse of a stellar cluster and (iii) direct collapse of gas.

What are the typical black hole seed masses for each of these channels?

One of the highest redshift black hole candidates detected by the James Webb Space Telescope, GN-z11, has an estimated black hole mass of $10^6\,\mathrm{M}_\odot$ at redshift $z = 10.6$, corresponding to a cosmic time of $t \sim 0.43$ Gyr. If this black hole was seeded at $z = 25$ ($t \sim 0.128$ Gyr), and accretes and emits at some fraction of Eddington luminosity, $f_{\rm Edd}$, roughly calculate if, and under which conditions, it could have originated from any of the three seeding channels discussed above.

Discuss the physical meaning of your findings, which seeding channels are more likely and why, and whether any of the seeding channels can be ruled out.

[Hint: You may assume that the radiative efficiency is 0.1 and the characteristic Eddington or Salpeter timescale is 0.45 Gyr. The following may be useful: $\exp(6) \sim 400$ and $\exp(12) \sim 1.8 \cdot 10^5$.]

(b) Consider a spherically symmetric halo, which is well described by the singular isothermal sphere with a velocity dispersion σ. It contains a fraction $f_{\rm gas}$ which is composed of gas and dust mixture. A supermassive black hole at the centre of the halo emits radiation at a constant luminosity L and couples to the gas via radiation pressure on dust. Supposing that radiation pressure sweeps up all intervening gas into a thin shell of radius R, mass $M_{\rm sh}(R)$ and optical depth $\tau(R)$, write down the general equation for the rate of change of momentum of the shell and explain what each term means.

Now, ignoring the external gas pressure, express the radius of the shell in units of the transparency radius R_τ, i.e. $R' = R/R_\tau$, where R_τ is the radius for which the UV optical depth $\tau_{\rm UV} = 1$. Also express time in units of the dynamical time, i.e. $t' = t/t_\tau$, where $t_\tau = R_\tau/(\sqrt{2}\sigma)$. Show that the equation for the rate of change of momentum of the shell can be expressed in dimensionless form as:

$$\frac{d}{dt'}\left[R'\frac{dR'}{dt'}\right] = (1 - e^{-\tau_{\rm UV}} + \tau_{\rm IR})\xi - 1\,, \tag{1}$$

where $\tau_{\rm IR}$ is the optical depth in the infrared, $\xi = L/L_{\rm crit}$, with $L_{\rm crit} = 4 f_{\rm gas} c\sigma^4/G$, where c is the speed of light and G is the gravitational constant.

Assuming a single-scattering approximation, i.e. the Eddington wind assumption, show that equation (1) admits expanding solutions of the form:

$$\left(R'\right)^2 = (\xi - 1)\left(t'\right)^2 + 2\dot{R}'_0 R'_0 t' + \left(R'_0\right)^2 . \tag{2}$$

Here R'_0 is the initial shell launch radius normalised by R_τ.

In the same regime show that there is a solution such that the shell expands at a constant speed given by $\dot{R} = \sqrt{2(\xi - 1)}\,\sigma$.

[QUESTION CONTINUES ON THE NEXT PAGE]

Now, in the regime where the optical depth is given by $\tau = (1 - e^{-\tau_{\mathrm{UV}}})$ determine the $\tau(R^{'})$ dependence on $R^{'}$ in order to show that there is a radius beyond which the shell will decelerate. What is the physical meaning of shell stalling in this regime?

Using this stalling radius calculate the time when the shell stalls, to show that there is a range of mildly super-critical shells with $1 < \xi < 1 + \epsilon$, such that the time of stalling is shorter if they are more super-critical. Hence, estimate ϵ and explain the physical meaning of this result.

Consider now the multi-scattering regime to show:

$$\dot{R}^{'} = \left(\frac{4\kappa_{\mathrm{IR}}\xi}{3\kappa_{\mathrm{UV}}}\right)^{1/3} (t^{'})^{-1/3}\,, \tag{3}$$

where κ_{UV} and κ_{IR}, are the opacities in the UV and IR, respectfully. What is the physical meaning for this result, and what can you deduce about the propagation of radiation pressure-driven outflows to large radii in the host halo?

END OF PAPER