UNIVERSITY OF CAMBRIDGE

MAMA/346, NST3AS/346, MAAS/346, PCAYM3/346

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 9:00pm to 12:00pm

PAPER 346

FORMATION OF GALAXIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Observations of the Milky Way across the electromagnetic spectrum reveal a wide range of physical processes operating within the Galaxy. All-sky and panoramic maps at wavelengths from radio to gamma-rays show markedly different morphologies, reflecting the distribution of stars, gas, dust, magnetic fields and high-energy particles.

In each of the following wavebands, identify the dominant emission mechanisms and the Galactic components being traced, commenting on whether the emission primarily probes density, temperature, composition or energetic particle populations:

(i) radio continuum emission at frequencies of order 10^8–10^9 Hz,

(ii) the 21 cm hyperfine transition of neutral hydrogen,

(iii) rotational line emission from CO molecules,

(iv) mid- and far-infrared emission,

(v) near-infrared emission,

(vi) optical starlight,

(vii) X-ray emission,

(viii) gamma-ray emission.

Explain why the large-scale appearance of the Milky Way differs substantially between the radio, infrared and optical bands, even though the underlying stellar and gaseous distributions are the same.

The Galactic plane appears extremely narrow in some wavebands and much broader in others. Explain physically (a) what determines the vertical thickness of emission in different tracers and (b) why synchrotron radio emission and gamma-ray emission extend to much higher Galactic latitudes than molecular or infrared emission.

The infrared and CO maps of the Milky Way show strong spatial correlations. Explain (a) the physical origin of this correlation and (b) how these tracers may be used to infer the Galactic star formation rate and its radial dependence.

In a short paragraph or two, discuss how multiwavelength observations allow the three-dimensional structure of the Milky Way bar to be reconstructed, despite our location within the Galactic disk.

2 The Zel'dovich approximation provides a Lagrangian description of gravitational instability in an expanding universe. The comoving Eulerian position $\mathbf{x}(\mathbf{q}, t)$ of a fluid element with Lagrangian coordinate $\mathbf{q}$ is given

$$\mathbf{x}(\mathbf{q}, t) = \mathbf{q} + D(t)\,\mathbf{s}(\mathbf{q}),$$

where $D(t)$ is the linear growth factor and $\mathbf{s}(\mathbf{q})$ is the displacement field. Show that mass conservation implies

$$1 + \delta(\mathbf{x}, t) = \left|\det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right)\right|^{-1},$$

where δ is the density contrast.

Now assume that the displacement field is irrotational

$$\mathbf{s}(\mathbf{q}) = -\nabla_q \Phi(\mathbf{q}),$$

where Φ is a scalar potential. Show that the deformation tensor

$$d_{ij} = \frac{\partial s_i}{\partial q_j}$$

may be diagonalised with real eigenvalues $-\lambda_1$, $-\lambda_2$, $-\lambda_3$.

Show that the density contrast may be written as

$$1 + \delta = \prod_{i=1}^{3} \frac{1}{1 - D(t)\lambda_i}.$$

Discuss the physical significance of the condition $1 - D(t)\lambda_i = 0$

Consider an initial perturbation for which only one eigenvalue is positive. Describe the resulting collapse geometry and relate it to the formation of large-scale structure.

Explain qualitatively how the Zel'dovich approximation predicts the emergence of a cosmic web composed of sheets, filaments and nodes.

Despite its simplicity, the Zel'dovich approximation remains remarkably accurate on large scales. Discuss the reasons for its success, and outline its main limitations.

Now consider a one-dimensional perturbation such that the displacement field is

$$s(q) = -A\sin(kq),$$

where A and k are positive constants. Write down the mapping $x(q, t)$ in the Zel'dovich approximation.

Show that the density contrast is given by

$$1 + \delta(x, t) = |1 - D(t)Ak\cos(kq)|^{-1}.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

Determine the earliest time at which shell crossing occurs.

Show that, at this time, the density formally diverges at isolated points and identify their locations.

By expanding the mapping $x(q,t)$ locally about a shell-crossing point, determine how the density scales with distance from the caustic.

3 The density profile of a dark matter halo is described by the Navarro–Frenk–White (NFW) model

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2},$$

where r_s is the scale radius and ρ_s is the characteristic density. Show that the enclosed mass of an NFW halo is

$$M(r) = 4\pi\rho_s r_s^3 \left[\ln\left(1+\frac{r}{r_s}\right) - \frac{r/r_s}{1+r/r_s}\right].$$

Show that the circular velocity profile may be written as

$$v_c^2(r) = 4\pi G\rho_s r_s^2 \frac{\ln(1+x) - \frac{x}{1+x}}{x}, \qquad x \equiv \frac{r}{r_s}.$$

Determine the asymptotic behaviour of the circular velocity for $r \ll r_s$ and for $r \gg r_s$.

Comment on how this behaviour compares with the rotation curves of typical spiral galaxies.

A spiral galaxy has an observed maximum circular velocity $v_{\max} = 220\ \mathrm{km\,s^{-1}}$ at a radius $r_{\max} = 20$ kpc. If the baryonic content can be neglected, estimate the scale radius r_s and the characteristic density ρ_s. (You may assume that the circular velocity curve of an NFW model reaches a maximum at $r_{\max} \approx 2.16\, r_s$. You may give your answer either numerically or in symbolic form).

Observed rotation curves of dwarf and low-surface-brightness galaxies often rise more slowly in their central regions than predicted by the NFW model. Explain this discrepancy and briefly discuss a possible physical explanation.

Now let us consider a galaxy which has both an NFW halo and a baryonic component described by a thin exponential disk with surface density

$$\Sigma(R) = \Sigma_0 \exp\left(-R/R_d\right),$$

where R_d is the disk scalelength and Σ_0 is the central surface density.

Show that the total mass of the disk is $M_{\mathrm{disk}} = 2\pi\Sigma_0 R_d^2$.

The galaxy has a disk scale length $R_d = 3$ kpc and an observed flat rotation speed of $v_c \approx 220\ \mathrm{km\,s^{-1}}$ for $R \gtrsim 10$ kpc. Estimate the disk mass required for the disk contribution to account for the observed rotation speed near $R \approx 7$ kpc. (You may assume that the peak of the rotation curve of an exponential disk is $v_{\mathrm{disk,max}} \approx 0.88(GM_{\mathrm{disk}}/R_d)^{1/2}$ and occurs at $R \approx 2.2R_d$).

Explain qualitatively how the halo parameters (r_s, ρ_s) must change if the disk mass is now increased.

What is a *maximum disk* model?

[QUESTION CONTINUES ON THE NEXT PAGE]

What observational or theoretical arguments can be used to support or disfavour maximum disks?

Explain why rotation curve decomposition into disk and halo components is inherently degenerate.

Give two methods by which this degeneracy may be reduced.

4 Consider a satellite galaxy on a circular orbit around a host galaxy. Both systems have power-law density profiles

$$\rho_{h,s}(r) = A_{h,s}\, r^{-\gamma}.$$

The normalisation coefficient for the satellite A_s is much lower than that for the host with A_h so that $A_s \ll A_h$. Find the enclosed mass, circular velocity and gravitational potentials of the models.

The satellite galaxy is subject to two effects:

(i) dynamical friction, described by Chandrasekhar's formula

$$\frac{dv}{dt} = -4\pi G^2 M_s \rho_h \frac{\ln \Lambda}{v^2} K,$$

where $\ln \Lambda$ and K are constants, whilst v is the satellite's speed;

(ii) tidal stripping, which removes material outside the tidal radius r_t. The satellite density profile is assumed unchanged inside r_t and sharply truncated beyond it.

Determine the tidal radius r_t of the satellite as a function of its orbital radius r_o (assuming a circular orbit).

Show that the tidal radius may also be written as

$$\frac{r_t}{r_o} \sim \left(\frac{M_s(< r_t)}{M_h(< r_o)} \right)^{1/3},$$

where $M_s(< x)$ and $M_h(< x)$ are the enclosed masses of satellite and halo respectively within radius x.

Now let $\gamma = 1$ for both satellite and halo. Assuming that the satellite evolves through a sequence of circular orbits, show that the differential equation for the time evolution of the orbital radius is

$$\frac{dr_o}{dt} = -C\sqrt{r_o},$$

where C is a positive constant.

Show that the satellite reaches zero radius in a finite time t_{merge}.

Now suppose that satellite mass stays constant, so we ignore the effects of tidal stripping entirely. Show that the satellite reaches zero radius in a time $t_{\text{merge}}/5$.

Explain this result physically.

Now suppose that the host galaxy has a singular isothermal density profile with $\rho \propto r^{-2}$ but the satellite density still follows the r^{-1} profile truncated at the tidal radius. Show that, under the combined effects of tidal stripping and dynamical friction, the satellite cannot reach zero radius in finite time.

Provide a physical interpretation of this result.

Explain any observational consequences?

END OF PAPER