MAMA/343, NST3AS/343, MAAS/343

MAT3
MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 2:00pm to 4:00pm

PAPER 343

QUANTUM ENTANGLEMENT IN MANY-BODY PHYSICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of a quantum channel and define the associated Krauss operators. What properties do the Krauss operators have to satisfy? Are they unique? What is the differential equation that you get in the limit of infinitesimal changes?

(b) What are the conditions on the Krauss operators such that this quantum channel is strictly contractive and has a unique fixed point?

(c) Write down a specific quantum channel for a 2-qubit system that is contractive and has the state $\rho = |\psi^-\rangle\langle\psi^-|$ with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt(2)$ as a fixed point. What is the second largest (in magnitude) eigenvalue of this channel?

(d) Write down the purification of this channel when it is applied multiple times. Demonstrate that this gives rise to a Matrix Product State (MPS). How unique is this MPS? Give necessary and sufficient conditions for the quantum channel to have a unique fixed point.

2

(a) What is the monogamy property of entanglement? Explain why this is relevant to quantum cryptography.

(b) Given a system of $N \to \infty$ qubits and the so-called XY-Hamiltonian

$$\hat{H} = \sum_{ij}^{N} X_i X_j + Y_i Y_j$$

where the sum over i and j runs over all sites and thus the interaction term acts over any pair of qubits, determine the ground state energy density (i.e. total energy of ground state divided by the total number of pairs). Is the energy density smaller in the case of 3 qubits?

(c) The monogamy property of quantum entanglement is clearly relevant for the study of quantum many-body systems. Argue why the monogamy property of entanglement gives rise to area laws for the entanglement entropy in ground states of quantum spin systems.

(d) Prove that matrix product states (MPS) and projected entangled pair states (PEPS) with fixed bond dimensions χ satisfy an area law for the entanglement entropy.

3 The cluster state is defined by the MPS tensors ($d = 2$, $D = 2$)

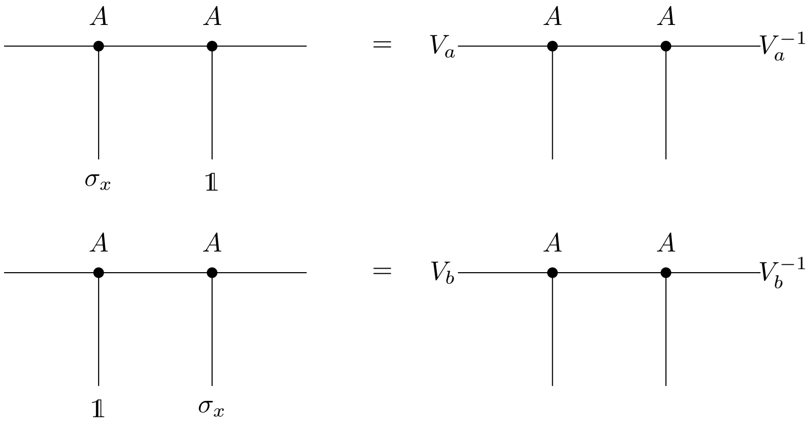
$$A^0 = |0\rangle\langle +| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \ , \qquad A^1 = |1\rangle\langle -| = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \ .$$

(a) The *injectivity length* of an MPS tensor is the smallest length j for which blocks of length j are injective, i.e. for which the products of matrices $A^{i_1}A^{i_2}\cdots A^{i_j}$, $i_k \in \{0,\ldots,d-1\}$, span the space of $D\times D$ matrices. What is the injectivity length of the cluster state MPS tensor?

(b) (i) Find unitary matrices V_a, V_b such that, for all i_1, i_2,

$$\sum_{i_1',i_2'} \langle i_1 i_2|\sigma_x \otimes \mathbb{1}|i_1' i_2'\rangle \cdot A^{i_1'}A^{i_2'} = V_a A^{i_1}A^{i_2}V_a^{-1} \ ,$$
$$\sum_{i_1',i_2'} \langle i_1 i_2|\mathbb{1} \otimes \sigma_x|i_1' i_2'\rangle \cdot A^{i_1'}A^{i_2'} = V_b A^{i_1}A^{i_2}V_b^{-1} \ .$$

These conditions can be expressed diagrammatically as



(ii) Use the existence of these matrices to argue that the periodic MPS of even length $2L$

$$|\psi\rangle = \sum_{i_1,\ldots,i_{2L}} \mathrm{Tr}[A^{i_1}A^{i_2}\cdots A^{i_{2L}}]|i_1,\ldots,i_{2L}\rangle \ .$$

has the following symmetries:

- A symmetry $\mathbb{Z}_2^{(a)}$ given by acting with σ_x simultaneously on all of the odd sites:

$$|i_1,\ldots,i_{2L}\rangle \mapsto (\sigma_x \otimes \mathbb{1} \otimes \sigma_x \otimes \mathbb{1} \otimes \cdots \sigma_x \otimes \mathbb{1})|i_1\cdots i_{2L}\rangle \ .$$

- A symmetry $\mathbb{Z}_2^{(b)}$ given by acting with σ_x simultaneously on all of the even sites:

$$|i_1,\ldots,i_{2L}\rangle \mapsto (\mathbb{1} \otimes \sigma_x \otimes \mathbb{1} \otimes \sigma_x \otimes \cdots \mathbb{1} \otimes \sigma_x)|i_1\cdots i_{2L}\rangle \ .$$

[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Together, these operations generate a $\mathbb{Z}_2^{(a)} \times \mathbb{Z}_2^{(b)}$ group of symmetries. Show that the cluster state has a nontrivial SPT invariant, i.e. that the matrices V_a, V_b define a nontrivial projective representation.

(iv) Consider the MPS $|\psi\rangle$ from part (ii) twisted by the boundary condition B:

$$|\psi(B)\rangle = \sum_{i_1,\ldots,i_\ell} \mathrm{Tr}[BA^{i_1}A^{i_2}\cdots A^{i_\ell}]|i_1,\ldots,i_\ell\rangle \ .$$

For each of the matrices $B = \mathbb{1}, V_a, V_b, V_aV_b$, compute the charges $\theta_B(g)$, $g = a, b$, of the twisted state:

$$\mathbb{Z}_2^{(g)} : |\psi(B)\rangle \mapsto \theta(g)|\psi(B)\rangle \ .$$

(c) The *parent Hamiltonian* of interaction length ℓ of an MPS tensor is the sum

$$H = \sum_k \mathbb{1}_{1,\ldots,k-1} \otimes h_{k,\ldots,k+\ell} \otimes \mathbb{1}_{k+\ell+1,\ldots,N} \ ,$$

where each local term h is the orthogonal projector onto the space of states of the form

$$|v(B)\rangle = \sum_{i_1,\ldots,i_\ell} \mathrm{Tr}[BA^{i_1}A^{i_2}\cdots A^{i_\ell}]|i_1,\ldots,i_\ell\rangle \ ,$$

for some matrix B. Let e_{ij} denote the matrix with entry 1 in position (ij) and 0 elsewhere. Show that

$$|v(e_{00})\rangle = |000\rangle + |001\rangle + |010\rangle - |011\rangle \ ,$$

and compute similar expressions for $|v(e_{10})\rangle$, $|v(e_{01})\rangle$, $|v(e_{11})\rangle$. For the $\ell = 3$ cluster state MPS tensor, h projects onto the span of these four states. Argue that it is therefore given by

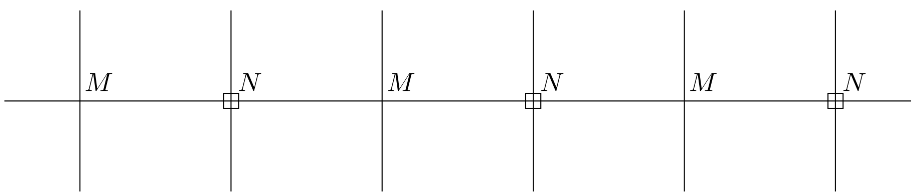
$$h = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - \sigma_z \otimes \sigma_x \otimes \sigma_z) \ .$$

Show that each term in H commutes with the others and therefore that they are frustration free. Show that the $\mathbb{Z}_2^{(a)} \times \mathbb{Z}_2^{(b)}$ symmetry commutes with each term in H.

(d) (i) Consider an MPO with different tensors on even and odd sites

$$\mathcal{O} = \sum_{\substack{i_1,\ldots,i_{2L} \\ i'_1,\ldots,i'_{2L}}} \mathrm{Tr}[M^{i_1i'_1}N^{i_2i'_2}M^{i_3i'_3}N^{i_4i'_4}\cdots M^{i_{2L-1}i'_{2L-1}}N^{i_{2L}i'_{2L}}]|i_1i_2\cdots i_{2L}\rangle\langle i'_1i'_2\cdots i'_{2L}| \ .$$

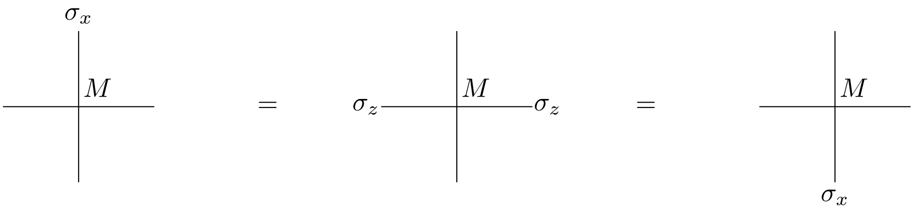
It can be expressed diagrammatically as shown:

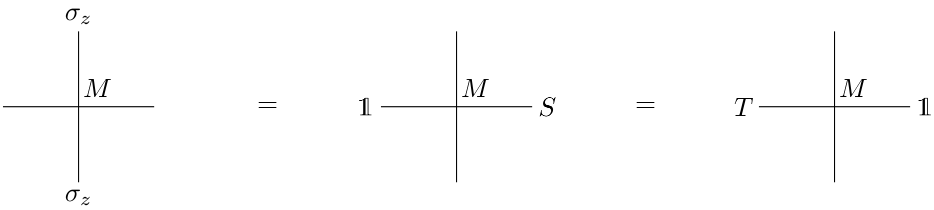


The tensors M and N are defined in the eigenbasis $|\pm\rangle$ of σ_x as follows:

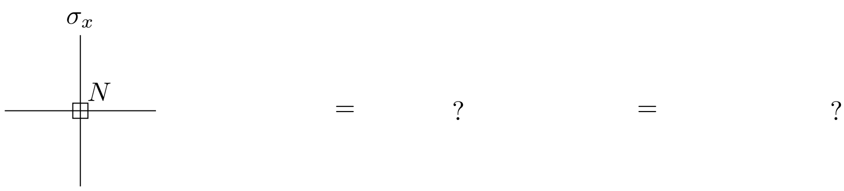
$$M^{++} = \mathbb{1} \ , \qquad M^{+-} = 0 \ , \qquad M^{-+} = 0 \ , \qquad M^{--} = \sigma_x$$
$$N^{++} = \mathbb{1} \ , \qquad N^{+-} = 0 \ , \qquad N^{-+} = 0 \ , \qquad N^{--} = \sigma_z$$

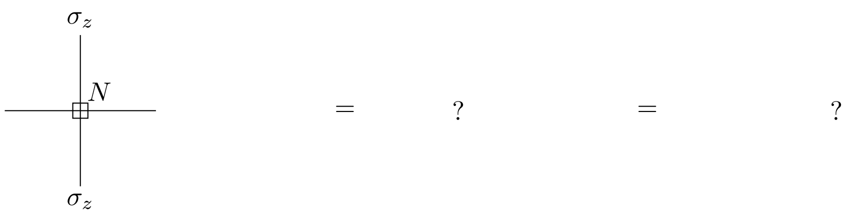
In the following diagrams, which express the symmetries of M, find the matrices S and T:





Give the analogous diagrams for N:





(ii) Use these relations to find the *dual Hamiltonian*, i.e. the operator H' such that

$$\mathcal{O}H = H'\mathcal{O} \ .$$

Describe the ground states of H'. Does the dual Hamiltonian exhibit SPT order, spontaneous symmetry breaking, neither, or both?

4

(a) The 2-dimensional Kramers-Wannier transformation has the form of a Projected Entangled Pair Operator (PEPO) and maps a quantum spin system defined on the vertices of a graph to a dual spin system defined on the edges of that graph. Make a graphical tensor network picture of this PEPO.

(b) Using diagrammatic arguments, or otherwise, show that the toric code Hamiltonian is the Kramers-Wannier dual of the Ising model defined on the square lattice.

(c) Define the symmetry of the original Ising system. What is the dual (one-form) symmetry of the toric code? Explain how the duality operator has only support on the corresponding symmetric subspaces, and discuss the consequence for the ground state degeneracy.

(d) What are the possible excitations in the toric code? What are their statistics? Give brief justifications for your answers.

END OF PAPER