UNIVERSITY OF CAMBRIDGE

MAMA/342, MAT3/342, MAPY/2/TQM, NST3PHY/2/TQM

MAT3
MATHEMATICAL TRIPOS Part III

Friday 12 June 2026 2:00pm to 4:00pm

PAPER 342

TOPOLOGICAL QUANTUM MATTER

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a two-dimensional topological phase described by the Chern-Simons (CS) Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi\hbar} K_{IJ} \varepsilon^{\alpha\beta\gamma} a_{I\alpha} \partial_\beta a_{J\gamma} - j_I^\alpha a_{I\alpha},$$

where the CS matrix K is an $n \times n$ symmetric, invertible matrix of integers, $a_{I\alpha}$ are CS gauge fields, and $j_I^\alpha = \sum_i j_{iI}^\alpha$ is the quasiparticle 3-current, with $j_{iI}^0 = q_I \delta[\mathbf{x} - \mathbf{x}_i(t)]$ and $\mathbf{j}_{iI} = (j_{iI}^1, j_{iI}^2) = \dot{\mathbf{x}}_i j_{iI}^0$, where the i^{th} quasiparticle is at position $\mathbf{x}_i$. (Summation over repeated Greek and uppercase Latin indices is implied.)

(a) Use the classical equations of motion (EOMs) for $a_{I\alpha}$ to show that encircling a quasiparticle of charges q_I with another quasiparticle of charges p_I yields a phase $\exp(2i\theta_{pq})$ with $\theta_{pq} = \pi \mathbf{p} \cdot K^{-1} \mathbf{q}$, where $(\mathbf{q})_J = q_J$ for the n-component vector $\mathbf{q}$, and similarly for $\mathbf{p}$. [In evaluating $\exp(2i\theta_{pq})$, you may use the shortcut of substituting the EOMs back into $\mathcal{L}$.]

(b) Consider the gauge transformation $a_{I\alpha} \to a_{I\alpha} + \partial_\alpha \Lambda_I$ with scalar functions Λ_I. Show that $\mathcal{L}$ is gauge invariant up to a total derivative. (You may use that $\partial_\alpha j_I^\alpha = 0$.)

(c) Suppose that the system is on a torus, allowing large gauge transformations where Λ_I is not single-valued but advances by $2\pi\hbar$ along a non-contractible loop. Show that the gauge invariance of $\exp(iS_c/\hbar)$, where $S_\text{c} = -\int d^3x \, j_I^\alpha a_{I\alpha}$, implies $\mathbf{q} \in \mathbb{Z}^n$.

(d) For $\mathbf{q}' = W\mathbf{q}$ with $W \in \mathrm{SL}(n, \mathbb{Z})$, the group of invertible integer matrices with unit determinant, show that if $\mathbf{q} \in \mathbb{Z}^n$, then $\mathbf{q}' \in \mathbb{Z}^n$, and that if $\mathbf{q}$ ranges over the whole $\mathbb{Z}^n$, then so does $\mathbf{q}'$. By requiring $j_I^\alpha a_{I\alpha}$ to be invariant, derive how $a_{I\alpha}$ transforms under W.

(e) Show that under W in (d), the CS matrix transforms as $K \to K' = WKW^\mathrm{T}$, where W^T is the transpose of W. Hence verify that K' is a symmetric, invertible integer matrix. Show, furthermore, that the CS theories with K and K' have the same set of $\exp(2i\theta_{pq})$ and the same torus ground-state degeneracy $|\det K| = |\det K'|$.

(f) The matrices K and K' in (e) show that the same system may be equivalently described by different CS matrices. Even without constructing W explicitly, the equivalence of two such theories can be established by comparing their torus ground-state degeneracies and braiding properties. Inspect these features to determine which of the CS matrices below describe the surface code:

$$K_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

[Hint: instead of e and m as the basic anyons, one can use f and m, where f is the result from fusing e and m. The f and m particles are mutual semions, while f has fermion self-statistics.]

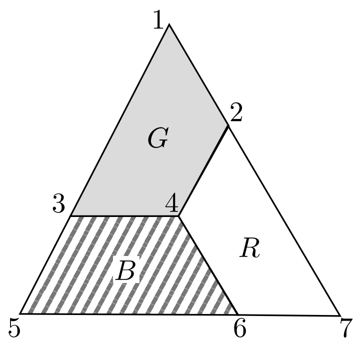
2 This question focuses on the Steane code and its generalisation to a color code.

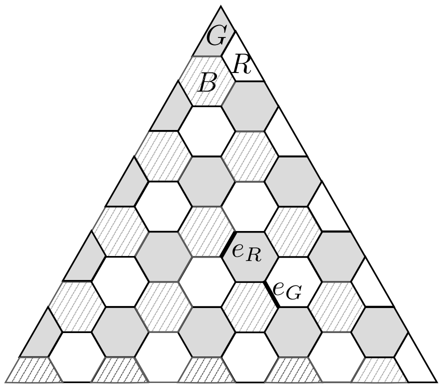
(a) State the Knill-Laflamme (KL) conditions for stabilizer codes, and use them to establish that a stabilizer code with distance $d \geqslant 2t+1$ can correct Pauli errors on any combination of qubits, provided that at most t qubits may suffer errors.

The 7-qubit Steane code has stabilizer generators (see also the left panel below)

$$S_i^{(X)}: X_1X_2X_3X_4,\ X_3X_4X_5X_6,\ X_2X_4X_6X_7; \quad S_i^{(Z)}: Z_1Z_2Z_3Z_4,\ Z_3Z_4Z_5Z_6,\ Z_2Z_4Z_6Z_7.$$

(b) Show that the code encodes one logical qubit and has distance $d=3$. Find an error set with single- and two-qubit Pauli errors that violates the KL conditions. From this set, find two errors with the same syndrome that illustrate why quantum error correction may fail when both one- and two-qubit Pauli errors can occur.





The Steane code (left) is the smallest instance of a triangular color code (right). Qubits are at the vertices, and the plaquettes P are labeled R (shown empty), G (shown gray), and B (shown striped). There is an X-type and Z-type stabilizer generator for each plaquette,

$$S_P^{(X)} = \prod_{j\in P} X_j, \quad S_P^{(Z)} = \prod_{j\in P} Z_j.$$

The color code Hamiltonian is $H = -\sum_P [S_P^{(X)} + S_P^{(Z)}]$, where we sum over all plaquettes.

(c) Show that the ground state $|\psi\rangle$ of H is a $+1$ eigenstate of all $S_P^{(X)}$ and $S_P^{(Z)}$. Show that, starting from $|\psi\rangle$, the application of $X_{e_R} = \prod_{j\in e_R} X_j$ for an "R edge" e_R connecting two R plaquettes (see the right panel above for an example) creates excitations at these R plaquettes. Call these excitations rx particles. Show that applying $X_{e_R'}$ along successive R edges e_R' moves the rx particles further apart.

(d) We can define ry, rz, gx, gy, gz, and bx, by, bz particles analogously; for example, gz is created by applying $Z_{e_G} = \prod_{j\in e_G} Z_j$ for a "G edge" e_G connecting two G plaquettes. Show that rx and gz particles are mutual semions.

(e) Show that the bottom boundary and the top corner of the triangular color code above condense gx, gy, and gz. Hence establish the code's distance, and provide examples for logical X and Z operators.

3 This question is on fermions, topological equivalence, and Majorana zero modes (MZMs).

(a) Show that any quadratic fermion Hamiltonian $H = \sum_{ij} h_{ij} a_i^\dagger a_j + \frac{1}{2}\Delta_{ij} a_i^\dagger a_j^\dagger + \frac{1}{2}\Delta_{ij}^* a_j a_i$ can be written, up to an additive constant, as $H = (i/4)\sum_{jk} A_{jk} c_j c_k$ with an even-dimensional real antisymmetric matrix A and Majorana fermions $c_{2j-1} = a_j + a_j^\dagger$, $\quad c_{2j} = -i(a_j - a_j^\dagger)$. Henceforth we take A to be of size $2M \times 2M$.

(b) The matrix A in (a) can be decomposed as $A = O^T \boldsymbol{\varepsilon} O$ where $\boldsymbol{\varepsilon} = \oplus_{j=1}^M \begin{pmatrix} 0 & \varepsilon_j \\ -\varepsilon_j & 0 \end{pmatrix}$ with $\varepsilon_j \geqslant 0$ and O is an orthogonal matrix. Hence show that a quadratic fermion Hamiltonian can be written in the "Majorana diagonal" form $H = \frac{1}{2}\sum_{j=1}^M \varepsilon_j i \chi_{2j-1}\chi_{2j}$ (up to an additive constant), where χ_i are Majorana fermions.

The rest of the question studies the Kitaev chain, with Hamiltonian $H = \sum_j (h_j - \mu a_j^\dagger a_j)$, where

$$h_j = -t(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + \Delta(a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger), \quad t, \Delta > 0,$$

or, in momentum space,

$$H = \frac{1}{2}\int_{-\pi}^{\pi} dk\, (\tilde{a}_k^\dagger, \tilde{a}_{-k}) H_{\mathrm{BdG}}(k) \begin{pmatrix} \tilde{a}_k \\ \tilde{a}_{-k}^\dagger \end{pmatrix}, \quad H_{\mathrm{BdG}}(k) = \begin{pmatrix} -2t\cos(k) - \mu & 2i\Delta\sin(k) \\ -2i\Delta\sin(k) & 2t\cos(k) + \mu \end{pmatrix}.$$

(c) Suppose $t = \Delta$ and $\mu = 0$, and denote the corresponding Hamiltonian as H_0. By expressing h_j using the Majorana fermions c_i from (a), show that H_0 is Majorana diagonal.

(d) Define the topological equivalence of local gapped Hamiltonians. Show that H is topologically equivalent to H_0 for $-2t < \mu < 2t$ (and $t, \Delta > 0$ arbitrary). Show how Hamiltonians with such parameters are distinguished from those with $\mu < -2t$ (and $\Delta, t > 0$ arbitrary) through topological properties of the mapping $k \to H_{\mathrm{BdG}}(k)$.

(e) Define MZMs. Consider H_0 with open boundary conditions, i.e., $H_0 = \sum_{j=1}^{M-1} h_j$ with $t = \Delta$ and $\mu = 0$, and show that c_1 and c_{2M} are MZMs.

(f) Consider again open boundary conditions. Using the local-unitary equivalence of ground states, show that for H topologically equivalent to H_0, there exist Majorana fermions γ_1 and γ_{2M} that are exponentially localised to sites 1 and M, respectively, and preserve the ground space of H (i.e., if $|\psi\rangle$ is in the subspace spanned by the two lowest-energy states of H, then so are $\gamma_1|\psi\rangle$ and $\gamma_{2M}|\psi\rangle$). Use that H is a quadratic fermion Hamiltonian to show that γ_1 and γ_{2M} are MZMs. [Hint: the local unitary U satisfies $U c_i U^\dagger = \sum_j f_{ji} c_j = \sum_j g_{ji}\chi_j$ with real f_{ji} and g_{ji}, where the χ_j are defined as in (b).]

END OF PAPER