UNIVERSITY OF CAMBRIDGE

MAMA/337, NST3AS/337, MAAS/337

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 16 June 2026 2:00pm to 4:00pm

PAPER 337

APPLICATIONS OF QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The retarded Green's function of an operator A at temperature T and frequency ω is given by

$$G^R(\omega) = -i \int_0^\infty dt\, e^{i\omega t}\, \mathrm{Tr}\left(e^{-H/T}[A(t), A(0)]\right) .$$

Here H is the Hamiltonian and there is no spatial dependence in this question.

(i) Argue that $G^R(\omega)$ is analytic in the upper half frequency plane.

(ii) Show that $\omega\, \mathrm{Im}\, G^R(\omega) \leqslant 0$, for ω real.

Consider the imaginary-time-ordered correlation function for the same operator A

$$C(\tau) = \begin{cases} \mathrm{Tr}\left(e^{-H/T} A(\tau)A(0)\right) & \tau > 0 \\ \mathrm{Tr}\left(e^{-H/T} A(0)A(\tau)\right) & \tau < 0 \end{cases}$$

(iii) Briefly explain why path integrals produce time-ordered correlation functions.

(iv) Show that $C(\tau)$ can be taken to be periodic with period $1/T$.

(v) Define the Matsubara frequencies $i\omega_n$ and thereby define the Fourier transformed correlation function $G(i\omega_n)$.

(vi) Show that

$$G(i\omega_n) = -\int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\mathrm{Im}\, G^R(\Omega)}{\Omega - i\omega_n} .$$

(vii) Consider the Drude response

$$G^R(\omega) = \frac{\sigma_o}{1 - i\omega t_o} . \qquad (*)$$

What condition or conditions must the constants t_o and σ_o obey? What is the physical interpretation of t_o?

(viii) Obtain $G(i\omega_n)$ corresponding to the Drude response $(*)$.

2 This question has two unrelated parts.

(a) The gap equation for the $O(N)$ model in three space dimensions is

$$\int \frac{d^3k}{(2\pi)^3} T \sum_{\omega_n} \frac{1}{v^2k^2 + \omega_n^2 + i\lambda} = \frac{1}{vg} \, .$$

We look for solutions $i\lambda = m^2$.

(i) Explain briefly the physical meaning of v, g and λ in the gap equation, in relation to antiferromagnetic spin waves. Briefly explain the role of the large N limit in obtaining the gap equation. Briefly explain why it is called the gap equation.

(ii) Obtain the zero temperature limit of the gap equation as a four-dimensional integral over d^4p. Introducing any UV or IR cutoffs that you may need, perform the integral. [You may find the substitution $x = p^2$ helpful. You may introduce 4-dimensional cutoffs.] Thereby obtain the critical coupling $g_{\rm c}$ at which spontaneous symmetry breaking occurs.

(iii) Obtain an implicit equation relating $g, g_{\rm c}$ and m. From this equation, obtain the behaviour of the gap m as $g \to g_{\rm c}$. Does it obey a simple quantum critical scaling?

(b) A ferromagnetic spin chain, in one spatial dimension, with a so-called easy-plane anisotropy has the following long-wavelength Langrangian density

$$\mathcal{L} = S(1 + \cos\theta)\partial_t \phi - u\left[(\partial_x\theta)^2 + \sin^2\theta(\partial_x\phi)^2\right] - K\cos^2\theta \, . \tag{1}$$

Here $u > 0$ and $K > 0$.

(i) By considering how the first term in the Lagrangian contributes to the path integral over closed trajectories, argue that the paramater S must be quantised.

(ii) Obtain the quadratic Lagrangian for the fluctuations $\theta = \frac{\pi}{2} - \delta\theta$ and $\phi = \delta\phi$. By solving the equations of motion following from the quadratic Lagrangian obtain the dispersion relation $\omega(k)$ for the spin waves. Consider the limits of large and small k of the dispersion and explain how these limits are related to the physics of conventional spin waves and superfluids.

END OF PAPER