UNIVERSITY OF CAMBRIDGE

MAMA/336, NST3AS/336, MAAS/336

MAT3

MATHEMATICAL TRIPOS Part III

Friday 5 June 2026 2:00pm to 4:00pm

PAPER 336

PERTURBATION METHODS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Legendre polynomial $P_n(\mu)$ for positive integer n can be represented by

$$P_n(\mu) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(z^2-1)^n}{(z-\mu)^{n+1}}dz$$

where C is a simple closed contour enclosing the point μ and $z=1$, but not the point $z=-1$, in the anticlockwise sense.

Use the method of steepest descents to calculate the leading order term in the asymptotic expansion of $P_n(\mu)$ as $n \to +\infty$ for the cases;

1. $\mu = \cos\theta$, $\theta \in (0, \pi/2)$
2. $\mu = \cosh\xi$, $\xi \in \mathbb{R}$

You may use the following without proof:

$$\frac{d}{dz}\left(\frac{z^2 - 2\cos\theta + 1}{(z^2-1)(z-\cos\theta)}\right)\Bigg|_{z=e^{i\theta}} = -ie^{-i\theta}\csc\theta$$

$$\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$$

2

The function $f(r)$ satisfies the equation

$$f_{rr} + \frac{3}{2r} f_r + \epsilon f f_r = 0 \quad \text{in} \quad r \geqslant 1$$

where $\epsilon > 0$ is a constant.

$f(r)$ is subject to the conditions

$$f = 0 \quad \text{at} \quad r = 1 \quad \text{and} \quad f \to 1 \quad \text{as} \quad r \to \infty.$$

Use the method of matched asymptotic expansions to solve the above equation.

Your solution should include;

1. A four-term asymptotic expansion for $f(r)$ as $\epsilon \to 0$ for $r = O(1)$.
2. A three-term asymptotic expansion for $F(\rho) = f(\epsilon r)$ as $\epsilon \to 0$, for $\rho = \epsilon r = O(1)$.
3. Matching these two expansions using any matching method.

You may use the fact that the general solution $y(x)$ with $y \to 0$ as $x \to \infty$ of

$$(x^{3/2} e^x y')' = E_{3/2}(x) = \int_x^\infty t^{-3/2} e^{-t} dt$$

is $y(x) = BE_{3/2}(x) + Y(x)$, where $Y(x)$ is a known function that you do *not* need to find this function explicitly, and B is an arbitrary constant.

You may also use without proof that as $x \to 0$ the function $y(x) = BE_{3/2}(x) + Y(x)$ has the following asymptotic expansion;

$$y = B(2x^{-1/2} - 2\sqrt{\pi} + 2x^{1/2} + O(x^{3/2})) + (4\log(x) + C + O(x^{1/2})).$$

Here C is a known numerical constant you may use in your solution (you do *not* need to find C explicitly).

3

Consider the coupled differential equations defined for $t \geqslant 0$

$$\frac{d^2v}{dt^2} + \omega^2 v = 2\epsilon \frac{d}{dt}\left[(1-Z)v\right]$$

$$\tau \frac{dZ}{dt} + Z = v^2$$

with ω, τ, and ϵ constants, with $0 < \epsilon \ll 1$, and ω and τ are $O(1)$.

Use the method of multiple scales to calculate the complete form of the leading-order term in the real asymptotic solutions for v and Z, correct up to times $t = O(\epsilon^{-1})$.

You are encouraged to use the form $A(T)e^{i\omega t}$ for your solutions.

You are encouraged to relabel complex constants into a manageable form, for example;

$$\frac{3 + 4i\omega\tau}{1 + 2i\omega\tau} = \alpha + i\beta$$

$$\frac{2\tau - 2i\omega\tau^2}{1 - 2i\omega} = \gamma + i\delta$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ do not need to be explicitly derived, and may be used in your final solution.

END OF PAPER