UNIVERSITY OF CAMBRIDGE

MAMA/334, NST3AS/334, MAAS/334

MAT3

MATHEMATICAL TRIPOS Part III

Friday 12 June 2026 9:00am to 11:00am

PAPER 334

BIOLOGICAL FLOWS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 We consider a two-dimensional model of microcirculation in which blood vessels are modelled as parallel, straight rigid surfaces separated by a distance $2h$ and subject to pressure-driven (Poiseuille) flow.

(a) (i) We first consider the design of blood vessels. Explain briefly Murray's law in three dimensions.

(ii) Adapt the modelling approach of Murray to the two-dimensional case, deriving the two-dimensional version of Murray's law.

(iii) Explain how your result can be interpreted in terms of shear stress, as was done in three dimensions.

(b) (i) We now consider the effective viscosity of blood as a suspension. What is the Fahraeus-Lindqvist effect and what is its origin?

(ii) We model blood as a suspension in which cells form a region of constant viscosity μ occupying the central part of the vessel, with a cell-free plasma layer (of constant viscosity μ_w) of fixed thickness δ near the surfaces. Defining the effective viscosity μ_{eff} as the viscosity of a single Newtonian fluid filling the entire gap $2h$ that would produce the same flow rate under the same pressure gradient, determine μ_{eff} in terms of μ, μ_w and δ/h.

(iii) Do the results agree with intuition in the limits $\delta = 0$ and $\delta = h$?

(c) (i) What is the Zweifach-Fung (ZF) effect?

(ii) To provide a local physical mechanism for ZF, we consider the branching of an upstream vessel into two daughter vessels, each of constant width $2h$: One vessel has flow rate Q_1 and the other has $Q_2 > Q_1$. We consider the fate of a red blood cell (of typical radius a) sitting symmetrically on the separatrix streamline (i.e. the streamline splitting the flow into the two daughter vessels) just before the branching point.

Using the fact that the flow in each daughter vessel is Poiseuille, derive order-of-magnitude estimates for the pressure gradient in each daughter vessel as a function of their flow rates.

(iii) Estimate the shear stresses, σ_1 and σ_2, exerted on either side of the cell arising from the flow in each daughter vessel. Deduce the net moment exerted on cell as a function of the difference $Q_1 - Q_2$. What does this predict for the red blood cell and is this consistent with ZF?

2 This question is about a model of cytoplasmic streaming in plant cells.

(a) We model an elongated plant cell as a cylinder of radius R aligned with the z-axis, with cytoplasmic flow driven by prescribed axial motion of the boundary. In cylindrical coordinates (r, θ, z), the axial velocity at $r = R$ is given as $u_z(R, \theta) = U$ for $0 \leqslant \theta < \pi$ and $-U$ for $-\pi \leqslant \theta < 0$, where U is a constant. The flow is independent of z. The fluid is Newtonian with constant viscosity μ, satisfies the incompressible Stokes equations, and no external pressure gradient is imposed.

(i) Assuming a unidirectional solution $\mathbf{u} = u_z(r, \theta)\mathbf{e}_z$, show that u_z satisfies Laplace's equation in the disk $0 \leqslant r < R$.

(ii) Solve for $u_z(r, \theta)$ subject to regularity at $r = 0$ and the boundary condition at $r = R$. Express your answer as a Fourier series.

(iii) Evaluate the series to obtain a closed-form solution for $u_z(r, \theta)$.

(b) We next study the transport of nutrients. To simplify the geometry, we consider a two-dimensional version where flow occurs between parallel rigid surfaces (oriented along x) separated by a distance $2h$ along y. The length of the cell along x is L. The flow is unidirectional $\mathbf{u} = u(y)\mathbf{e}_x$ with boundary conditions $u = \pm U$ at $y = \pm h$, with solution $u(y) = Uy/h$. Nutrients with concentration $c(x, y, t)$ diffuse with diffusivity D, are advected by the flow, and satisfy $\partial c/\partial y = 0$ at $y = \pm h$.

(i) Define the two relevant Péclet numbers. By comparing them with the aspect ratio L/h, determine the limit in which the Taylor–dispersion regime is obtained.

(ii) By considering the full advection-diffusion equation, splitting c into a mean $(\bar{c})$ and fluctuating value (c'), use scaling arguments to derive the balance in the Taylor-dispersion regime between advection of mean longitudinal concentration gradients and diffusion of transverse gradients.

(iii) Compute the Taylor dispersion coefficient (i.e. the flow-induced diffusivity) as a function of U, h and D.

Hint: Laplace's equation in polar coordinates (r, θ) is given by

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

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