UNIVERSITY OF CAMBRIDGE

MAMA/333, NST3AS/333, MAAS/333, NST3QC/FDC, MAAMM2/FDC

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 9:00am to 12:00pm

PAPER 333

FLUID DYNAMICS OF CLIMATE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Linearised shallow water equations

The linearized shallow water equations on an $f-$plane, for a layer of fluid above a flat bottom, are given by:

$$u_t - fv = -g\eta_x, \tag{1}$$

$$v_t + fu = -g\eta_y, \tag{2}$$

$$\eta_t + H_0(u_x + v_y) = 0. \tag{3}$$

(a) Explain the meaning of each of the quantities appearing in these equations and set out the assumptions made in deriving them from the Navier-Stokes equation in a rotating frame. State the equations for geostrophic balance. Write down an expression for the linearised disturbance to the potential vorticity, q.

(b) Consider an initial disturbance to the surface height, $\eta(x, y, t)$, of a fluid initially at rest, such that $\eta(x, y, 0) = \eta_0 \operatorname{sgn}(y)$. (Note: $\operatorname{sgn}(y) = y/|y|$ for $y \neq 0$ and 0 for $y = 0$.) Obtain a form for the final surface height and velocities after geostrophic adjustment and identify a lengthscale that naturally arises in the problem. Briefly describe how the adjustment to steady state is achieved.

For the remainder of the question, consider the addition of a solid boundary (e.g., a coast) running from North to South at $x = 0$ and the behaviour in the region $x > 0$. The boundary condition requires that $u = 0$ at $x = 0$.

(c) In the presence of the boundary, the above equations have a Kelvin wave solution in which $u = 0$ everywhere, and η and v are bounded as $x \to \infty$. In this solution, η may be written in the form $\eta(x, y, t) = G(x)F(y+\gamma t)$. Determine the value of γ and the form of $G(x)$. Explain how these waves behave differently in the Northern and the Southern Hemispheres. Calculate the linearized disturbance to the potential vorticity associated with the Kelvin waves.

(d) Now consider the geostrophic adjustment from the initial condition given in (b), but in the presence of the boundary. Assume that $f > 0$. Without detailed calculations, explain what aspects of the mathematical problem, determining the adjusted state, must be changed to take into account the presence of the boundary. What is the appropriate boundary condition on η and what is the role of the Kelvin wave in determining this boundary condition? Sketch the streamlines for the steady state flow with the boundary.

2 Ocean surface height

Consider wind-driven ocean circulation in a rectangular basin in the domain $0 \leqslant x \leqslant L$, $0 \leqslant y \leqslant W$ on a β-plane. The fluid has constant mean depth H_0 and constant density ρ. The dynamics are described by the forced shallow-water quasi-geostrophic (QG) equation

$$\frac{\partial P_g}{\partial t} + J(\psi, P_g) = \frac{1}{\rho H_0^2}(\nabla \times \boldsymbol{\tau}) \cdot \hat{\mathbf{z}}, \tag{1}$$

where

$$P_g = \left(\frac{f}{f_0} + \frac{\nabla^2 \psi}{f_0} - \frac{f_0 \psi}{g H_0}\right), \tag{2}$$

is the shallow water QG potential vorticity, ψ is the QG streamfunction, $f = f_0 + \beta y$ is the Coriolis parameter, H_0 is the mean fluid height, g is the gravitational acceleration, $J(A, B) = A_x B_y - A_y B_x$, and $\hat{\mathbf{z}}$ is the unit vector in the local vertical direction. The wind stress is purely zonal and varies with latitude:

$$\boldsymbol{\tau} = \tau(y)\hat{\mathbf{x}}, \tag{3}$$

with $\tau(0) = \tau(W) = 0$.

(a) Starting from equations (1) and (2) derive an expression for Sverdrup balance written in terms of the QG streamfunction ψ, the wind stress curl, and other parameters in the problem. State clearly any additional assumptions that are needed.

(b) Derive an expression for the free-surface height $\eta_{\text{int}}(x, y)$ associated with flow in Sverdrup balance for the form of the wind stress given in equation (3). Hence, find the change in surface height between the western and eastern sides of the basin that would result if the flow is in Sverdrup balance everywhere. Using shallow water geostrophic balance, find an expression for the north/south transport in the basin,

$$T = H_0 \int_0^L v dx,$$

in terms of the surface height, η.

(c) Now, consider the addition of linear bottom drag with coefficient γ so that the shallow water QG equation becomes

$$\frac{\partial P_g}{\partial t} + J(\psi, P_g) = -\gamma \nabla^2 \psi + \frac{1}{\rho H_0^2}(\nabla \times \boldsymbol{\tau}) \cdot \hat{\mathbf{z}}.$$

Using a scaling analysis or otherwise, determine a scaling for the width of the western boundary current, δ, in terms of the other physical parameters.

(d) Applying no normal flow boundary conditions on the sides of the basin, find the difference in free surface height within the western boundary current,

$$\Delta \eta_{WBC} = \eta(x = \delta, y) - \eta(x = 0, y).$$

You may assume that the width of the boundary layer is small compared to the width of the basin, i.e. $\delta \ll L$.

(e) Obtain an expression for the mean velocity in the western boundary layer,

$$\overline{v} = \frac{1}{\delta} \int_0^{\delta} v dx.$$

Comment on the dependence of $\overline{v}$ on γ and explain the presence and strength of the flow in the western boundary layer in terms of physical principles.

3 Oceanic Rossby Waves

The 3D quasi-geostrophic equation (3D QG) evolution equation can be written

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \tag{1}$$

where $J(A,B) = A_x B_y - A_y B_x$ and the 3D QG potential vorticity is

$$q = \psi_{xx} + \psi_{yy} + \left(\frac{f_0^2}{N^2}\psi_z\right)_z + \beta y, \tag{2}$$

where $f = f_0 + \beta y$ is the Coriolis parameter, N is the buoyancy frequency which may be assumed to be constant, and ψ is the geostrophic streamfunction.

(a) Consider a basic state where $\psi = 0$ in the domain $-H \leqslant z \leqslant 0$ with flat, rigid boundaries at $z = -H$ and $z = 0$. Small amplitude perturbations of the form

$$\psi' = \mathrm{Re}\left(\hat{\psi}(z)e^{i(kx+ly-\omega t)}\right), \tag{3}$$

are added.

(i) Express the impermeable boundary condition at $z = -H$ and $z = 0$ in terms of ψ'.

(ii) Derive the vertical structure equation for $\hat{\psi}$ and the corresponding dispersion relation for $\omega(k,l)$.

(b) Now consider the problem described in part (i), but with the rigid surface at $z = 0$ replaced with a free surface located at $z = \eta(x,y,t)$.

(i) Assuming that the surface displacement is small, but non-zero, derive the boundary condition to be applied at $z = 0$ in terms of ψ'. You may assume that the pressure and surface height are related through hydrostatic balance such that

$$p(x,y,z=0,t) = p_a + \rho_0 g \eta, \tag{4}$$

where p_a is the constant atmospheric pressure, ρ_0 is a constant reference density, and g is the gravitational acceleration.

(ii) Consider separable solutions of the form $\psi' = A(x,y,t)\hat{\psi}(z)$. For small surface displacements the form of $\hat{\psi}(z)$ can be approximated by

$$\hat{\psi}(z) = \cos\left(\frac{n\pi z}{H}\right), \tag{5}$$

where $n = 1, 2, \ldots$ is the vertical mode number. Find an expression for the surface height, η, in terms of the amplitude $A(x,y,t)$ and other parameters in the problem.

(iii) Find an expression for the maximum vertical displacement of density contours in the ocean interior. If the depth of the ocean is $H = 3000$ m and $N = 10^{-3}$ s^{-1}, estimate the maximum displacement of density surfaces in the ocean interior if the maximum free surface height displacement is $\eta = 1$ cm.

4 Equatorial Waves

The Boussinesq primitive equations on an equatorial β-plane, making the 'long-wave approximation' may be written as

$$u_t - \beta y v = -\phi_x, \tag{1}$$

$$\beta y u = -\phi_y, \tag{2}$$

$$u_x + v_y + w_z = 0, \tag{3}$$

$$\sigma_t + N^2 w = 0, \tag{4}$$

$$\phi_z = \sigma, \tag{5}$$

where (x, y, z) are Cartesian coordinates respectively in the eastward, northward and upward directions and t is time. The quantity N is constant.

(a) Give the physical meaning of each of these equations and the variables that appear in them.

For parts (b)-(d) below, consider solutions of the form $u = \Re(\hat{u}(y)e^{i(kx+mz-\omega t)})$, with analogous forms for v, w, ϕ and σ, with k, m and ω real constants and where the function $\hat{u}(y)$ and its analogues decay as $|y| \to \infty$. Assume that $k > 0$. These solutions will be referred to as *wave solutions*.

(b) First seek solutions with $v = 0$. Derive the dispersion relation

$$\omega = Nk/|m| \tag{6}$$

and the corresponding form of $\hat{\phi}(y)$. Explain carefully why $\omega = -Nk|m|$ is *not* an acceptable solution.

(c) For solutions with v non-zero, derive expressions for $\hat{u}(y)$ and $\hat{\phi}(y)$ in terms of $\hat{v}(y)$ and hence show that

$$\hat{v}_{yy} - \frac{\beta^2 y^2 m^2}{N^2}\hat{v} = \frac{k\beta}{\omega}\hat{v}.$$

Deduce the dispersion relation

$$\omega = -\frac{Nk}{(2n+1)|m|} \qquad (n = 1, 2, \dots) \quad .$$

[You may use the fact that the eigenvalue problem defined by the ordinary differential equation $V_{\xi\xi} - \xi^2 V = \lambda V$ with $|V| \to 0$ as $|\xi| \to \infty$, has eigenvalues $\lambda_n = -(2n+1)$, $n = 0, 1, 2, \dots$ with eigenfunctions $V_n(\xi) = H_n(\xi)e^{-\frac{1}{2}\xi^2}$, where $H_n(\xi)$ is the nth Hermite polynomial.]

Explain why your calculation excludes the choice $n = 0$. (In the remainder of the question you may simply assume that $n = 0$ is not allowed.)

For each n, denote the solutions for $\hat{v}(y)$ and $\hat{\phi}(y)$ by, respectively, $\hat{v}_n(y)$ and $\hat{\phi}_n(y)$. Give the explicit form of $\hat{v}_n(y)$ and, in terms of $\hat{v}_n(y)$, the corresponding form of $\hat{\phi}_n(y)$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(d) Consider a problem with fluid in the region $z > 0$ and a lower boundary at $z = 0$. Disturbances are forced by specifying w on $z = 0$ as $w = W(y)\cos(k(x - Ct))$, where C is a constant and $W(y) \to 0$ as $|y| \to \infty$. An approach to solving this problem is to identify all of the wave solutions identified above that potentially match the boundary conditions, to superpose those solutions in a series with unknown coefficients and then to determine the coefficients such that the sum precisely matches the boundary condition.

Write down the form as, functions of x, y, z and t, of the relevant sum of wave solutions for (a) $C > 0$ and (b) $C < 0$ (use the $\hat{v}_n(y)$ and $\hat{\phi}_n(y)$ notation where convenient) and in each case give the condition that the boundary condition is matched.

Consider the case $C > 0$. Why it is clear that most forms of $W(y)$ cannot be matched by a sum of wave solutions? What can you say about the physical nature of the remaining part of the solution in this case? [Hint: think about the corresponding mid-latitude problem as described by quasi-geostrophic theory.]

END OF PAPER