UNIVERSITY OF CAMBRIDGE

MAMA/332, NST3AS/332, MAAS/332, NST3QC/FDSE, MAAMM2/FDSE

MAT3

MATHEMATICAL TRIPOS Part III

Wednesay 10 June 2026 2:00pm to 5:00pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
Questions 1 and 2 carry half the credit of questions 3 and 4.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 An ice shell grows at the surface of a planetary satellite over an interior global ocean. The thickness of the ice shell is small compared to the planetary radius, so that the dynamics are only a function of the vertical coordinate z (pointing downwards), with $z = 0$ on the surface of the ice shell.

(a) Construct an energy balance at the surface of the ice shell taking account of the incoming solar radiation F_r, outgoing short and long wavelength radiation and the conductive heat flux F_c from the ice shell. The outgoing long wavelength radiation may be modelled as $F_o = \epsilon\sigma T_s^4$, where ϵ is the planetary emissivity, σ is the Stefan-Boltzmann constant and T_s is the surface temperature. The albedo of the ice shell is α. Take the conductivity of the shell to be temperature-dependent, $k(T) = a/T$, where a is a constant and T is the temperature. Draw a schematic of the energy balance at the surface, and derive an implicit expression for the surface temperature.

(b) Consider the conservation of thermal energy within the ice shell in which ρ is the density of ice, c_p is the specific heat capacity of the ice, and the thermal conductivity is temperature-dependent, $k(T) = a/T$ (as in part a). The base of the ice shell, at $z = h$, is at the melting temperature, $T(h) = T_m$, and there is an associated heat flux F_w from the ocean beneath the ice shell. Determine the steady-state temperature profile in the shell, $T(z)$, and use energy conservation at the base of the ice shell (the Stefan condition) to determine the steady-state thickness of the ice shell h as a function of F_w, T_m and T_s.

(c) Using the quasi-steady temperature profile (derived in part b), the energy balance at the surface and Stefan's law at the base of the shell find approximate expressions for the surface temperature in the limit of a thin shell, $h \ll a/F_w$, and a thick shell, $h \gg a/F_w$. Hence, or otherwise, estimate the growth rate of the ice shell as a function of time and sketch a plot of $h(t)$.

(d) On long timescales the ice shell may undergo solid-state deformation, and thus behave as a viscous fluid (similar to mantle convection on Earth). Describe how you might estimate if and when the shell might convect. Discuss if convection is more or less likely on a larger planetary body.

2 A lake of water is at far-field temperature T_∞ and at $t = 0$ the surface temperature is fixed to a constant $T_b < T_m$, where T_m is the equilibrium melting temperature. A layer of ice grows from the surface of thickness $h(t)$. The temperature in the ice and the water satisfies a thermal diffusion equation with diffusivity κ (you should assume equal material properties in the ice and water).

(a) Draw a schematic of the ice-covered lake, including the thermal field, your coordinate axes, and any applicable boundary conditions.

(b) Derive dimensional self-similar solutions for the thermal field in the ice and the water beneath. Note that the ice thickness may be written as $h = 2\lambda\sqrt{\kappa t}$. Use the Stefan condition to find an implicit equation for the constant λ, where L is the latent heat of fusion and c_p is the specific heat capacity.

(c) The ice grows into a salt-stratified lower layer of far-field concentration C_0 at a depth H below the surface. The ice rejects salt as it freezes so remains pure ($C_{\text{solid}} = 0$), and the interface between the ice and salty layer remains planar and horizontal. At times much after the ice grows into the salty layer, find a self-similar solution for the salt concentration in the water as the ice grows, written in terms of the diffusivity of salt D, the interfacial concentration $C(h) = C_i$ and the far-field concentration C_0, and where $\epsilon = \sqrt{D/\kappa}$. You do not need to derive an explicit expression for C_i, but should discuss how C_i might be determined.

(d) The melting temperature of ice is a function of salt concentration and given approximately by $T_m = -mC$ for constant $m > 0$. If the diffusivity of salt is much less than the diffusivity of heat, $\epsilon \ll 1$, discuss how and where constitutional supercooling might develop. You should draw and annotate a phase diagram with the equilibrium melting line and a profile of (T, C) showing the trajectory through the liquid and solid phases both before and after the growth of ice encounters the salty layer.

The following mathematical definitions may be of use,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy, \quad erfc(x) = 1 - erf(x), \quad erf(0) = 0, \quad erf(\infty) = 1.$$

3 Consider the viscous flow of a two-dimensional ice cap on a flat, horizontal base at $z = 0$ with thickness $h(x,t)$, where x is the horizontal distance from the ice divide and t is time. The ice cap is formed by an accumulation and ablation rate

$$a(h, h^\star) = \begin{cases} \alpha, & h > h^\star, \\ -\alpha, & h < h^\star, \end{cases}$$

which here is taken as a constant $\pm\alpha$ with respect to the height above/below the snowline $h^\star$. The ice cap is long and thin, and the flow is driven by the gradients in hydrostatic pressure within the ice cap and limited by vertical shear within the ice, which has constant viscosity μ. The surface of the ice is stress free, and the ice cap slides along the bed with basal velocity

$$u_b = \beta \frac{\tau_b}{\rho g h}, \quad \text{where} \quad \tau_b = \mu \left.\frac{\partial u}{\partial z}\right|_{z=0}$$

is the basal shear stress and β is a positive constant.

(a) Draw a clearly labelled diagram of the ice cap, and derive explicit expressions for the horizontal velocity within the ice, $u(x,z,t)$, and the basal sliding speed u_b.

(b) Using an expression of local mass conservation, show that the evolution of the ice cap is governed by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(u_b h) = \frac{\rho g}{3\mu} \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x} \right) + a,$$

where ρ is the density of ice and g is the gravitational acceleration.

(c) In a stable climate a steady state exists in which net accumulation is balanced by net ablation. Determine the steady profile and extent of the two-dimensional ice cap by first scaling the evolution equation (using $h^\star$ as the vertical scale) subject to zero ice flux at the ice divide, $q(x = 0) = 0$, continuity of ice thickness h and ice flux where $h = h^\star$, and zero thickness and flux at the nose.

(d) Determine the dimensional conditions under which basal sliding dominates the ice flux, and find the (dimensional) steady-state extent of the ice cap in this limit.

(e) In the limit in which basal sliding dominates, consider the case of rapid climate change for which at $t = 0$ the current is uniformly melting, $h \leqslant h^\star$ for all x. Show that an analytical solution for the decay of the ice cap may be found. To do so you may wish to first find a self-similar solution for the spreading current, h_{ss}, for which $\alpha = 0$ and then look for solutions $h = h_{ss} + ct$, for which you should find the constant c. Hence, determine the dimensional timescale for the ice cap to fully melt away if at $t = 0$ $h(0,0) = h^\star$.

4 At the end of the last ice age, the rapid retreat of the Laurentide and Fennoscandian ice sheets left surface depressions which have since rebounded. Consider the time dependence of these surface depressions as governed by the slow, incompressible, viscous flow of the mantle described by

$$\nabla \cdot \mathbf{u} = 0,$$

$$0 = -\nabla p + \mu \nabla^2 \mathbf{u} - \rho g \hat{\mathbf{z}}.$$

Here $\mathbf{u}$ is the velocity, p is the pressure, μ is the constant viscosity of the mantle, ρ is the density of the mantle, g is the local gravitational acceleration and $\hat{\mathbf{z}}$ is the unit vector pointing upwards. Mantle velocities are driven by surface deformation and decay at depth, $\mathbf{u} \to 0$ as $z \to -\infty$. Consider a two-dimensional response, which may therefore be written in terms of a streamfunction ψ, such that the velocity $\mathbf{u} = (u, 0, w) = \nabla \times (\psi \hat{\mathbf{y}})$.

(a) Show that the streamfunction formulation automatically satisfies mass conservation and, by taking the curl of Stoke's equations, derive expressions for the the horizontal and vertical velocities and the pressure within the mantle.

(b) The deformed surface may be described by $\zeta(x,t) = \epsilon(t)e^{ikx}$, where ϵ is the amplitude and k is the horizontal wavenumber, and $\epsilon k \ll 1$. Construct expressions for the outward normal $\hat{\mathbf{n}}$ and tangent $\hat{\mathbf{t}}$ to the deformed surface in the (x, z)-plane.

(c) Consider the case in which the mantle is overlain by an elastic plate with bending stiffness B, such that the horizontal velocity tangent to the surface vanishes, and the normal stress is balanced by bending stresses within the plate,

$$\mathbf{u} \cdot \hat{\mathbf{t}} = 0 \quad \text{and} \quad \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} + B\nabla^4 \zeta = 0 \quad \text{on} \quad z = \zeta,$$

where $\boldsymbol{\sigma} = -p\mathbf{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the stress tensor and $\mathbf{I}$ is the identity tensor. Linearise the surface boundary conditions, and hence find expressions for the horizontal and vertical velocities at $z = 0$.

(d) Use the kinematic condition at the free surface to determine the time dependence of the surface depression. Plot the timescale for vertical motion as a function of the wavelength $\lambda = 2\pi/k$ describing the surface deformation and interpret your results, making particular reference to the effects of elasticity.

(e) The mantle is of finite depth H. Without accounting for the radial geometry of the Earth, describe (without direct calculation) how your analysis could be extended to account for the conditions at the base of the mantle, in which the mantle floats on the liquid iron inner core (the density of iron $\rho_c > \rho$).

END OF PAPER