UNIVERSITY OF CAMBRIDGE

MAMA/329, NST3AS/329, MAAS/329, NST3QC/SVF, MAAMM2/SVF

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 2:00pm to 5:00pm

PAPER 329

SLOW VISCOUS FLOW

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Starting with the Stokes equations (with no body force) and making the substitution $p = \nabla^2 \Pi$, derive the Papkovich–Neuber representation of Stokes flow in terms of two harmonic potentials $\boldsymbol{\Phi}$ and χ.

Use this representation to find the Stokes flow $\mathbf{u}$ due to a rigid sphere of radius a rotating with angular velocity $\boldsymbol{\Omega}$ in unbounded viscous fluid. Explain carefully the reasons for your choice of harmonic potential.

Give two different arguments for why the pressure due to the flow is zero.

Calculate the couple from the fluid on the sphere. Calculate also the value of $\partial \mathbf{u}/\partial r$ on $r = a$.

(b) Now consider a nearly spherical rigid particle with radius (in spherical polar coordinates) $r = a + \epsilon f(\theta, \phi)$, where $\epsilon \ll 1$. The particle is force-free but is moving under the influence of a fixed applied couple $\mathbf{G}$ in unbounded viscous fluid. The velocity of the particle $\mathbf{U} + \boldsymbol{\Omega} \wedge \mathbf{x}$ can be expanded as a regular perturbation series

$$\mathbf{U} = \mathbf{U}_0 + \epsilon \mathbf{U}_1 + O(\epsilon^2) \quad \text{and} \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_0 + \epsilon \boldsymbol{\Omega}_1 + O(\epsilon^2)$$

with similar expansions for the velocity field $\mathbf{u}$ and stress field $\boldsymbol{\sigma}$. What are the values of $\mathbf{U}_0$ and $\boldsymbol{\Omega}_0$?

What are the exact boundary conditions satisfied by $\mathbf{u}$ and $\boldsymbol{\sigma}$ on the surface S of the particle? By expanding these conditions in powers of ϵ, deduce that

$$\mathbf{u}_1 = \mathbf{U}_1 + \boldsymbol{\Omega}_1 \wedge \mathbf{x} + f \boldsymbol{\Omega}_0 \wedge \mathbf{x}/a - f \frac{\partial \mathbf{u}_0}{\partial r} \quad \text{on } r = a$$

and

$$\int_{r=a} \boldsymbol{\sigma}_1 \cdot \mathbf{n} \, dS = \int_{r=a} \mathbf{x} \wedge \boldsymbol{\sigma}_1 \cdot \mathbf{n} \, dS = \mathbf{0}.$$

Explain why there are no terms involving f and $\boldsymbol{\sigma}_0$ in the two integrals.

(c) Use the reciprocal theorem on $\mathbf{u}_1$ and a suitable test flow $\hat{\mathbf{u}}$ to obtain an equation for $\boldsymbol{\Omega}_1$. For the case of a slightly ellipsoidal particle with $f = a(\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n})$, where $\mathbf{D}$ is a constant, symmetric, traceless, second-rank tensor, show that

$$\boldsymbol{\Omega}_1 = \alpha \mathbf{D} \cdot \boldsymbol{\Omega}_0,$$

where the constant α should be calculated. Explain briefly why $\mathbf{U}_1 = \mathbf{0}$ for such a particle.

$\left[\textit{You may assume that} \displaystyle\int_{r=a} n_i n_j n_k n_l \, dS = \frac{4\pi a^2}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \right]$

2 Fluid of viscosity μ surrounds a very long axisymmetric fluid bubble of radius $r = a(z,t)$ (in cylindrical polar coordinates) and viscosity $\lambda\mu$, where $\lambda \ll 1$. The bubble radius a varies slowly with z, so that $|\partial a/\partial z| \ll 1$. The ends of the bubble are far enough away to be neglected, and the flow is driven only by surface tension.

Explaining any approximations made to the boundary conditions on $r = a$, show that the external flow is approximately radial and that the pressure $P(z,t)$ in the bubble is given by

$$P = \frac{2\mu}{a}\frac{\partial a}{\partial t} + \frac{\gamma}{a}\,, \tag{1}$$

where γ is the coefficient of surface tension.

(a) Assume that λ is sufficiently small that the pressure gradient inside the bubble can be neglected. Verify that both (1) and conservation of bubble volume are satisfied by a solution of the form $a(z,t) = f(t) + \sqrt{2}g(t)\sin(kz)$, where $f > \sqrt{2}g > 0$, provided

$$P(t) = \frac{\gamma f(t)}{\alpha_0}, \qquad \frac{df}{dt} = -\frac{\gamma}{2\mu}\frac{g^2(t)}{\alpha_0} \quad \text{and} \quad \frac{dg}{dt} = \frac{\gamma}{2\mu}\frac{f(t)g(t)}{\alpha_0}\,,$$

where α_0 is a constant to be determined in terms of $f(0)$ and $g(0)$.

Deduce the linear growth rate of a small long-wavelength sinusoidal disturbance to a cylindrical bubble of radius a_0. Show also that the minimum radius $a_{\min}(t)$ decreases monotonically to zero as the disturbance continues to evolve in the nonlinear regime. [*You do not need to solve the equations for f and g explicitly.*]

(b) You are given that the pressure gradient inside the bubble can only be neglected while $a_{\min}a_0k^2 \gg \lambda$ and so the assumption in part (a) becomes invalid for small enough $a_{\min}$. Using lubrication theory, show that the evolution is then given by

$$2a\frac{\partial a}{\partial t} = \frac{1}{8\lambda\mu}\frac{\partial}{\partial z}\left\{a^4\frac{\partial}{\partial z}\left(\frac{2\mu}{a}\frac{\partial a}{\partial t} + \frac{\gamma}{a}\right)\right\}. \tag{2}$$

Just before the bubble breaks at some time t^* and position z^*, there is a local similarity solution of (2) in which $a \propto (t^* - t)^p$ and $z - z^* \propto (t^* - t)^q$ as $t \to t^*$. Use scaling estimates of the different terms in (2) to find dimensionless similarity variables ζ and $A(\zeta)$, such that $A(\zeta)$ obeys a third-order ordinary differential equation, to be determined, that is free of any parameters. [*You are not required to solve this equation.*]

3

(a) A thin layer of viscous fluid of thickness $h(x,t)$ flows due to constant surface tension and gravity on a rigid plane that is inclined at angle θ to the horizontal. The plane moves with speed $-U$ in the x-direction, where x is a fixed downslope coordinate. Use the equations of lubrication theory to derive the evolution equation

$$\frac{\partial h}{\partial t} - U\frac{\partial h}{\partial x} + \frac{\rho g \sin\theta}{3\mu}\frac{\partial h^3}{\partial x} = \frac{1}{3\mu}\frac{\partial}{\partial x}\left(h^3\left\{\rho g\cos\theta\,\frac{\partial h}{\partial x} - \gamma\frac{\partial^3 h}{\partial x^3}\right\}\right).$$

Explain briefly why it is possible to impose a boundary condition $h \to h_0$ as $x \to -\infty$ if $U < \rho g h_0^2 \sin\theta/\mu$, but not if $U > \rho g h_0^2 \sin\theta/\mu$.

(b) Now consider steady flow of a thin layer of viscous fluid down a stationary vertical plane into a large pool of the same fluid. Far above the pool $h(x) \to h_0$, where h_0 is such that $\epsilon \equiv (\rho g h_0^2/\gamma)^{1/3} \ll 1$. Use part (a) to obtain the dimensionless equation

$$H^3 H''' = 1 - H^3 \tag{1}$$

for the layer thickness $H(X)$ above the pool, where H and X should be defined. What is the asymptotic form of $H - 1$ as $X \to -\infty$?

The rest of the question considers the asymptotic form of H near the pool as $\epsilon \to 0$.

(c) Use a scaling argument to estimate the dimensional length scale ℓ of a static meniscus where a stagnant pool of fluid meets a stationary vertical plane. How does h_0 compare to ℓ?

As $\epsilon \to 0$ the asymptotic solution to (1) must satisfy matching from the (dimensional) curvature of the thin fluid layer to the curvature, $(2\rho g/\gamma)^{1/2}$, at the edge of a static meniscus as it (nearly) meets the plane tangentially. Why is this reasonable? Show that

$$H'' \sim (\epsilon/2)^{-1/2} \tag{2}$$

in the matching region.

(d) The linearised behaviour in part (b) develops into a sequence of capillary waves whose amplitudes increase rapidly towards the pool. Assume that as $\epsilon \to 0$ those closest to the pool have crests with $H \gg 1$ and troughs with $H \ll 1$.

Deduce the dominant *physical* balance in (i) the crests and (ii) the troughs.

In parts (e) and (f) below, assume that each crest matches to the slope of the next trough and that each trough matches to the curvature of the next crest until the final trough matches to the meniscus through (2).

(e) Suppose you are given a solution $\phi(\xi)$ to the problem

$$\phi^3\phi''' = 1, \qquad \phi'' \to 0 \text{ as } \xi \to -\infty, \qquad \phi'' \to 1 \text{ as } \xi \to \infty.$$

(ϕ is unique to within an arbitrary origin.) Deduce an approximation to the shape $H(X)$ of the final trough in terms of $\phi(\xi)$. If $\phi(\xi) \sim -A\xi$ as $\xi \to -\infty$, show that the matching condition to the preceding crest is $H' \sim -A(\epsilon/2)^{-1/10}$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(f) Find the solution, $\psi(\eta), \eta_0$, to the problem

$$\psi''' = -1, \qquad \psi(0) = \psi'(0) = 0, \qquad \psi(\eta_0) = 0, \qquad \psi'(\eta_0) = -1.$$

Deduce an approximation to the shape of the final crest in terms of $\psi(\eta)$ and show that the matching condition to the penultimate trough is $H'' \sim (2A/3)^{1/2}(\epsilon/2)^{-1/20}$.

By comparison with part (e), deduce the asymptotic scaling of H with ϵ in the penultimate trough. [*You can omit purely numerical factors.*] Comment on the applicability of this asymptotic analysis as $\epsilon \to 0$ to experimental observations of the waves observed in practice near entry into the pool.

END OF PAPER