UNIVERSITY OF CAMBRIDGE

MAMA/327, NST3AS/327, MAAS/327

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 16 June 2026 9:00am to 11:00am

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define the space of *Schwartz functions* $\mathcal{S}(\mathbf{R}^n)$ and the space of *tempered distributions* $\mathcal{S}'(\mathbf{R}^n)$, defining the standard notion of convergence on each.

(b) Prove a linear form $u : \mathcal{S}(\mathbf{R}^n) \to \mathbf{C}$ belongs to $\mathcal{S}'(\mathbf{R}^n)$ iff it is sequentially continuous.

(c) Let $\{f_\alpha\} \subset C(\mathbf{R}^n)$ be of polynomial growth, i.e. for each α there exists constants $C_\alpha, M_\alpha \geqslant 0$ such that

$$|f_\alpha(x)| \leqslant C_\alpha \left(1 + |x|\right)^{M_\alpha}.$$

Show that for each $N \geqslant 0$, $\sum_{|\alpha| \leqslant N} D^\alpha f_\alpha \in \mathcal{S}'(\mathbf{R}^n)$.

(d) Show that every tempered distribution with compact support can be written as a finite sum of derivatives of bounded, continuous functions.

(e) Define $u \in C^\infty(\mathbf{R})$ by

$$u(x) = \exp\left(x^2 + \mathrm{i}\exp\left(x^4\right)\right).$$

Does this function define an element of $\mathcal{S}'(\mathbf{R})$? Justify your answer.

2

(a) Let $P : \mathbf{C}^n \to \mathbf{C}$ be non-zero polynomial of order N. Give an explicit construction of an $E \in \mathcal{D}'(\mathbf{R}^n)$ for which

$$P(D)E = \delta_0.$$

(b) Suppose now that $P(\lambda) = P_1(\lambda_1) \cdots P_n(\lambda_n)$ with $\deg(P_i) > 1$ for each i. Show that there is a fundamental solution of the form

$$(x_1, \ldots, x_n) \mapsto \begin{cases} \prod_{j=1}^n \sum_{i=1}^{N_j} \alpha_{ij}(x) e^{\beta_{ij} x_j}, & x_1, \ldots, x_n > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha_{ij}(x)$ are complex polynomials and β_{ij} are complex numbers.

(c) Using your result in (b), show that a fundamental solution for the operator

$$P(D) = D_1^2 D_2^2 + D_1^2 - D_2^2 - 1$$

is given by $E(x_1, x_2) = \sin(x_1)\sinh(x_2)$ on $x_1, x_2 > 0$ and zero elsewhere.

3

(a) What does it mean to say that $P(D)$ is *elliptic*? Show that if $P(D)$ is Nth order elliptic then $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$ for $|\lambda|$ sufficiently large.

(b) Define a *parametrix* of $P(D)$ and construct such a parametrix when $P(D)$ is elliptic.

(c) Let $X \subset \mathbf{R}^n$ be open. Let $Q(x, D)$ be an Mth order partial differential operator with smooth coefficients such that for $|\lambda| > R$ and for each compact $K \subset X$ we have

$$\inf_{x\in K} |Q(x,\lambda)| \gtrsim_K \langle\lambda\rangle^M .$$

The remaining part of the question explores the construction of a parametrix for Q.

(i) Define the space of *symbols* $\mathrm{Sym}(X, \mathbf{R}^n; N)$. If $c_i \in \mathrm{Sym}(X, \mathbf{R}^n; N_i)$, $i = 1, 2$ establish the following

(I) $D_x^\alpha D_\lambda^\beta c_i \in \mathrm{Sym}(X, \mathbf{R}^n; N_i - |\beta|)$,

(II) $c_1 c_2 \in \mathrm{Sym}(X, \mathbf{R}^n; N_1 + N_2)$,

(III) $c_1 + c_2 \in \mathrm{Sym}(X, \mathbf{R}^n; \max\{N_1, N_2\})$.

(ii) Let $a \in C^\infty(X \times \mathbf{R}^n)$, $a(x, \lambda) = 0$ on $|\lambda| < R$ and $a(x, \lambda) = 1/Q(x, \lambda)$ for $|\lambda| > R + 1$. Show that this defines a symbol and specify its order, N_Q.

(iii) The distribution $A_1 \in \mathcal{D}'(X)$ is defined by the oscillatory integral

$$A_1(x) = \frac{1}{(2\pi)^n} \int e^{\mathrm{i}\lambda\cdot x} a_1(x,\lambda)\, \mathrm{d}\lambda,$$

where $a_1 = a \in \mathrm{Sym}(X, \mathbf{R}^n; N_Q)$. Show that

$$Q(x,D)A_1(x) = \frac{1}{(2\pi)^n} \int e^{\mathrm{i}\lambda\cdot x} Q(x, \lambda + D_x) a_1(x,\lambda)\, \mathrm{d}\lambda.$$

(iv) Show that $Q(x, \lambda + D_x)a_1 = \sum_{|\alpha|\leqslant M} D_\lambda^\alpha Q(x,\lambda) D_x^\alpha a_1(x,\lambda)/\alpha!$ and deduce

$$Q(x, \lambda + D_x)a_1(x,\lambda) = 1 - b_1(x,\lambda),$$

where $b_1 \in \mathrm{Sym}(X, \mathbf{R}^n; -1)$.

(v) Find a correction $a_2 \in \mathrm{Sym}(X, \mathbf{R}^n; N_Q - 1)$ so that

$$Q(x, \lambda + D_x)\left[a_1(x,\lambda) + a_2(x,\lambda)\right] = 1 - b_2(x,\lambda),$$

where $b_2 \in \mathrm{Sym}(X, \mathbf{R}^n; -2)$.

(vi) Deduce that there exists an oscillatory integral $A_1 + \cdots + A_N \in \mathcal{D}'(X)$ such that

$$Q(x,D)(A_1 + \cdots + A_N) = \delta_0 - B_N,$$

where $B_N \in C(X)$ for N sufficiently large.

END OF PAPER