UNIVERSITY OF CAMBRIDGE

MAMA/325, NST3AS/325, MAAS/325

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 2:00pm to 4:00pm

PAPER 325

QUANTUM INFORMATION, FOUNDATIONS AND GRAVITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Explain carefully what is meant by the reduced density matrix of a subsystem of a multipartite quantum system.

(b) A "system" qubit whose Hilbert space is H_S undergoes a sequence of interactions with "environment" qubits whose (distinct) Hilbert spaces are H_{E_i} $(i = 1, 2, 3, \ldots)$ in sequential short time intervals. These produce unitary evolutions of the form U_{SE_i} where

$$\begin{aligned}
U_{SE_i} |0\rangle_S |0\rangle_i &= |0\rangle_S |0\rangle_i \ , \\
U_{SE_i} |1\rangle_S |1\rangle_i &= |1\rangle_S (\cos\epsilon |1\rangle_i - \sin\epsilon |0\rangle_i) \ , \\
U_{SE_i} |0\rangle_S |1\rangle_i &= |0\rangle_S |1\rangle_i \ , \\
U_{SE_i} |1\rangle_S |0\rangle_i &= |1\rangle_S (\cos\epsilon |0\rangle_i + \sin\epsilon |1\rangle_i) \ .
\end{aligned}$$

Here the $\{|0\rangle, |1\rangle\}$ states form orthonormal bases for the relevant Hilbert spaces and ϵ is small. All other interactions, including self-interactions, have zero Hamiltonian at all times. The system qubit is initially in state $\alpha |0\rangle_S + \beta |1\rangle_S$ and the environment qubits are initially in states $\frac{1}{\sqrt{2}}(|0\rangle_i + |1\rangle_i)$.

What is the reduced density matrix of the system before any interaction? What is the reduced density matrix of the system after n environmental interactions? Explain how this can be interpreted as a model of environmental decoherence. Explain carefully how the effect of decoherence can be seen by carrying out a measurement on the system qubit (only).

Would your results and interpretation change if the system instead underwent repeated interactions of the above form with a single environmental qubit, and no other interactions? If so, how? If not, why not?

2 Consider the following device that could hypothetically be realised by physics beyond quantum theory. The device comprises two boxes, A and B, which may be arbitrarily separated. Each box accepts as input a unit vector defining a positive axis direction in 3-dimensional space. Given an input, each box produces an output ± 1. The respective outputs x, y in general depend probabilistically on both inputs. If unit vectors $\underline{a}, \underline{b}$ are input to A, B respectively, then the output correlations, defined by the expectation value of the product of x and y, satisfy

$$E(xy) = f(\theta(\underline{a}, \underline{b}))\,.$$

Here $\theta(\underline{a}, \underline{b})$ is the angle between $\underline{a}$ and $\underline{b}$, so that $\cos(\theta(\underline{a}, \underline{b})) = \underline{a}.\underline{b}$, and the function f is defined by

$$f(\theta) = \begin{cases} 1 & \text{for } 0 \leqslant \theta \leqslant \pi/6\,, \\ (-(\theta - \pi/6) + (\pi/4 - \theta))(12/\pi) & \text{for } \pi/6 \leqslant \theta \leqslant \pi/4\,, \\ -1 & \text{for } \pi/4 \leqslant \theta \leqslant \pi/3\,, \\ ((\theta - \pi/3) - (2\pi/3 - \theta))(3/\pi) & \text{for } \pi/3 \leqslant \theta \leqslant 2\pi/3\,, \\ 1 & \text{for } 2\pi/3 \leqslant \theta \leqslant 3\pi/4\,, \\ (-(\theta - 3\pi/4) + (5\pi/6 - \theta))(12/\pi) & \text{for } 3\pi/4 \leqslant \theta \leqslant 5\pi/6\,, \\ -1 & \text{for } 5\pi/6 \leqslant \theta \leqslant \pi\,. \end{cases}$$

(a) State EPR's criterion for an element of physical reality. State the CHSH inequality.

(b) Show that this two-box device maximally violates the CHSH inequality for an appropriate choice of inputs, which you should identify.

(c) Show that, if such devices were found in nature, there could not be elements of physical reality $x(\underline{a}), y(\underline{b})$ pre-determining the outputs as functions of local inputs.

(d) Show that with appropriate choices of output probabilities the two-box device can be consistent with relativistic causality, i.e., does not allow superluminal signalling.

3 Write down the Schrödinger equation for two mass m particles interacting gravitationally. You may assume here, and throughout this question, that all other forces (including external gravitational fields) are negligible. Describe how the state evolves from an initial state in which particle i is in a superposition state of the form $\frac{1}{\sqrt{n}}(\sum_{a=1}^{n} \psi_{ai})$, where ψ_{ai} is localised around horizontal position x_{ai} for $i = 1, 2$, assuming that the ψ_{ai} remain localized around their respective horizontal positions (which are all distinct) throughout the evolution.

Suppose now that the separations $|x_{a1} - x_{a2}| = d$ are small and identical, while the other separations are sufficiently large that the gravitational potential between the masses in these locations is relatively negligible, and that the particles maintain all these relative separations throughout. By calculating the reduced density matrix of one particle, or otherwise, show that the system reaches a maximally entangled state. In the large n limit, with $m = 10^{-14}$kg and $d = 2 \times 10^{-4}$m, estimate (to within 20%) when it reaches that state.

Does the system remain entangled for all times $t > 0$? Justify your answer.

(You may take Newton's gravitational constant $G = 6.7 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and Planck's constant $h = 6.6 \times 10^{-34}$ m^{2}kg s^{-1}, with $\hbar = h/2\pi$.)

4 (a) Explain carefully how a hypothetical quantum state readout device that gives a complete classical description of a localized quantum state can be defined so that it produces a well defined output for any input in a given finite-dimensional Hilbert space and does not allow superluminal signalling. You may assume the device produces an output instantaneously with infinite precision and ignore the physical constraints that this assumption violates. You should describe how the device acts on a pure state, an improper mixed state and a proper mixed state of an input quantum system.

(b) Show carefully that extending quantum theory by such devices can give a logically consistent physical theory.

(c) The GHZ state of three qubits is

$$|\psi_{GHZ}\rangle = (\frac{1}{\sqrt{2}})(|0\rangle_A |0\rangle_B |0\rangle_C - |1\rangle_A |1\rangle_B |1\rangle_C)\,.$$

Suppose that space-time is 1+1 dimensional and that the qubits A, B, C are separated and kept at rest at positions $x = 0, 1, 2$ in an inertial frame, with space and time coordinates in units in which the speed of light $c = 1$. Suppose that at $t = 0$ the operator $(\sigma_z)_A$ is measured on A, at $t = 3$ the operator $(\sigma_x)_C$ is measured on C, and at $t = 5$ the operator $(\sigma_z)_B$ is measured on B.

What are the possible combinations of outcomes?

For any one possible outcome combination of your choice, describe what the output of a quantum state readout device would be for each of the qubits A, B, C at any time $t > 0$, identifying any points in time when any of the outputs change.

END OF PAPER