UNIVERSITY OF CAMBRIDGE

MAMA/323, NST3AS/323, MAAS/323, MAPY/2/QIT, NST3PHY/2/QIT

MAT3

MATHEMATICAL TRIPOS Part III

Friday 5 June 2026 2:00pm to 4:00pm

PAPER 323

QUANTUM INFORMATION THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Quantum channels

(a) (i) Give the definition of a quantum channel.

(ii) What is its Kraus representation?

(b) Let $T : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^{d'})$ be linear. Consider C its Choi matrix, given by

$$C := (T \otimes \mathrm{id}_d)(|\phi\rangle\langle\phi|) \tag{1}$$

where

$$|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |kk\rangle \tag{2}$$

is a maximally entangled state and $\{|k\rangle\}$ is an orthonormal basis. Show that if T is a quantum channel, then $C \geqslant 0$ and $\mathrm{Tr}_1[C] = \frac{\mathbb{1}}{d}$, for $\mathrm{Tr}_1[\cdot]$ the partial trace of the first system.

(c) Now take $d = d'$. A channel is called random unitary if

$$T(\rho) = \sum_k p_k U_k \rho U_k^* \,, \tag{3}$$

for probabilities $\{p_k\}$ satisfying $\sum\limits_k p_k = 1$ and a set of unitary operators $\{U_k\}$, where $*$ denotes Hermitian conjugation.

(i) Show that the Choi matrix of a random unitary channel is a convex combination of maximally entangled states.

(ii) Show that every random unitary channel is unital.

(iii) Show that the converse is not true as follows. Consider the Werner-Holevo channel $T : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$, with $d \geqslant 3$, given by

$$T(\rho) := \frac{1}{d-1}(\mathrm{Tr}[\rho]\mathbb{1} - \rho^T) \,, \tag{4}$$

where ρ^T is the transpose of ρ. Use (i) together with the fact that the Choi matrix associated to the transpose map is $\frac{1}{d}\mathbb{F}$, with $\mathbb{F}$ the flip operator.

2 Separability

Consider a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and an arbitrary density matrix $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$.

(a) (i) What is the definition of ρ_{AB} being separable or entangled?

(ii) State what the PPT criterion implies about entanglement or separability. For which dimensions does it provide a necessary and sufficient condition?

(b) Let A and B be two qubit systems. Consider the bipartite density matrix

$$\rho = \begin{pmatrix} a & 0 & 0 & x \\ 0 & b & y & 0 \\ 0 & y^* & c & 0 \\ x^* & 0 & 0 & d \end{pmatrix} \tag{1}$$

written in the computational basis, with $a, b, c, d \geqslant 0$, $a+b+c+d = 1$ and $x, y \in \mathbb{C}$.

(i) Derive necessary and sufficient conditions on x and y for ρ to be a valid density matrix.

(ii) Use the PPT criterion to determine conditions for x and y so that ρ is separable.

(c) For Hilbert spaces of general finite dimension, a channel T is entanglement breaking if for every bipartite state ρ_{AB},

$$(T \otimes \mathrm{id})(\rho_{AB}) \text{ is separable.} \tag{2}$$

(i) Show that any channel of the form

$$T(\rho) = \sum_k \mathrm{Tr}[M_k \rho]\sigma_k \tag{3}$$

with $\{M_k\}$ a POVM and $\{\sigma_k\}$ states is entanglement breaking.

(ii) Show that T is entanglement breaking if, and only if, its Choi matrix is separable.

In the above, state clearly any general results that you use.

Hint: Use in both points the Choi matrix of T, which is given by

$$C := (T \otimes \mathrm{id}_d)(|\phi\rangle\langle\phi|) \tag{4}$$

where

$$|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |kk\rangle \tag{5}$$

is a maximally entangled state and $\{|k\rangle\}$ is an orthonormal basis.

3 Hypothesis testing

(a) (i) Describe the framework of symmetric binary quantum hypothesis testing, with the two errors that may occur.

(ii) State and prove the Quantum Neyman-Pearson (Holevo-Helstrom) Theorem.

(b) Consider the ensemble of qubit states:

$$|\psi_k\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{2\pi i k/3}\sin\frac{\theta}{2}\,|1\rangle\ , \quad k = 0,1,2 \tag{1}$$

with equal probabilities

$$p_k = \frac{1}{3}\,. \tag{2}$$

(i) Compute the Pretty Good Measurement.

(ii) Show that, in this case, it is optimal. You may use without proofs general results from the course provided that they are correctly stated.

4 Entropies

The aim of this question is to show continuity of the conditional entropy.

(a) Define the von Neumann entropy of a state $\rho \in \mathcal{S}(\mathcal{H})$ in a finite-dimensional Hilbert space $\mathcal{H}$, and state its property of concavity.

(b) Using operator monotonicity of the logarithm, prove the property of quasi-convexity:

$$S(\rho) \leqslant pS(\rho_1) + (1-p)S(\rho_2) + H(p)\,, \tag{1}$$

for $\rho = p\rho_1 + (1-p)\rho_2$, with $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$ and $p \in [0,1]$, and where $H(p)$ is the binary entropy.

(c) Consider a bipartite space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$. Setting $\varepsilon := \frac{1}{2}\left\|\rho_{AB} - \sigma_{AB}\right\|_1$, follow the steps (i)-(v) below to prove that the conditional entropy satisfies:

$$|H(A|B)_\rho - H(A|B)_\sigma| \leqslant 2\varepsilon \log d_A + (1+\varepsilon) H\left(\frac{\varepsilon}{1+\varepsilon}\right)\,. \tag{2}$$

(i) Let $\Delta_{AB} = \frac{(\rho_{AB}-\sigma_{AB})_+}{\varepsilon}$. Check that it is an admissible state and define ω_{AB} as a convex combination of Δ_{AB} and σ_{AB} with coefficients $\frac{\varepsilon}{1+\varepsilon}$ and $\frac{1}{1+\varepsilon}$, respectively. Let Δ'_{AB} so that $\varepsilon\Delta'_{AB} = (1+\varepsilon)\omega_{AB} - \rho_{AB}$. Check that Δ'_{AB} is also a state.

(ii) Show that

$$\rho_{AB} \leqslant (1+\varepsilon)\omega_{AB}\,, \tag{3}$$

and analogously one has

$$\sigma_{AB} \leqslant (1+\varepsilon)\omega_{AB}\,. \tag{4}$$

(iii) Using the variational characterization for the conditional entropy

$$-H(A|B)_\omega = \min_{\xi_B} D(\omega_{AB}\,|\,\mathbb{1}_A \otimes \xi_B)\,, \tag{5}$$

and (b), show the following inequality:

$$H(A|B)_\omega \leqslant \frac{1}{1+\varepsilon} H(A|B)_\rho + \frac{\varepsilon}{1+\varepsilon} H(A|B)_{\Delta'} + H\left(\frac{\varepsilon}{1+\varepsilon}\right)\,. \tag{6}$$

(iv) Using (a), show

$$H(A|B)_\omega \geqslant \frac{1}{1+\varepsilon} H(A|B)_\sigma + \frac{\varepsilon}{1+\varepsilon} H(A|B)_\Delta\,. \tag{7}$$

(v) Combining (iii) and (iv), and showing that the conditional entropy of any state is bounded between $-\log d_A$ and $\log d_A$, conclude the proof of Eq. (2).

END OF PAPER