UNIVERSITY OF CAMBRIDGE

MAMA/321, NST3AS/321, MAAS/321, PCAYM3/321

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 16 June 2026 9:00am to 11:00pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Viscous evolution in Be stars

The evolution of a viscous accretion disc is governed by

$$\partial_t \rho + \frac{1}{r}\partial_r(r\rho u_r) + \frac{1}{r}\partial_\phi(\rho u_\phi) + \partial_z(\rho u_z) = 0,$$

$$\rho\left[\partial_t u_\phi + \frac{1}{r}u_r\partial_r(r u_\phi) + \frac{1}{r}u_\phi\partial_\phi u_\phi + u_z\partial_z u_\phi\right] = \frac{1}{r^2}\partial_r(r^2\Pi_{r\phi}) - \frac{1}{r}\partial_\phi\left(P - \Pi_{\phi\phi}\right) + \partial_z\Pi_{\phi z},$$

in cylindrical coordinates, where ρ is density, P is pressure, $\mathbf{u}$ is velocity, and $\mathbf{\Pi}$ is the viscous stress. Assume that $\rho\mathbf{u}$ and $\mathbf{\Pi}$ go to zero as $|z| \to \infty$.

(a) Assuming $u_\phi \approx r\Omega(r)$, derive the equations

$$\partial_t\Sigma + \frac{1}{2\pi r}\partial_r\mathcal{F} = 0, \qquad \mathcal{F}\frac{dh}{dr} + \partial_r\mathcal{G} = 0,$$

where Σ is surface density, $h = h(r)$ is the specific angular momentum, $\mathcal{F}$ is the radial mass flux, and $\mathcal{G}$ is the viscous torque, for which you should provide an expression in terms of $\Pi_{r\phi}$. Given that

$$\Pi_{r\phi} = \mu\left[r\partial_r\left(\frac{1}{r}u_\phi\right) + \frac{1}{r}\partial_\phi u_r\right],$$

where μ is the dynamical viscosity, show that $\mathcal{G} = -2\pi\bar{\nu}\Sigma r^3(d\Omega/dr)$, with $\bar{\nu}$ the mean viscosity. Finally, derive

$$\partial_t\left(2\pi r\Sigma h\right) + \partial_r\left(\mathcal{F}h + \mathcal{G}\right) = 0, \qquad (\dagger)$$

and explain the meaning of each term.

(b) A Be star spins near its break-up speed and can expel material from its equator. This material forms a disc, with inner radius r_{in}, which viscously spreads outward to a truncation radius r_{out}, where the star's radiation field removes the material in a wind.

Assume that the disc is in steady state and is fed mass at r_{in} at a constant rate $\dot{M}$. In addition, no torque is exerted at r_{out}. The disc may be modelled as Keplerian and its surfaces modelled as black bodies.

(i) Show that the steady state disc structure is given by

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left(\sqrt{\frac{r_{\text{out}}}{r}} - 1\right).$$

[*Note that in this scenario $\dot{M}$ is the outward mass transfer rate.*]

(ii) Suppose the torque at the inner boundary is fixed to be Θ by some magnetic process. Express the truncation radius in terms of Θ, and use equation ($\dagger$) to interpret this result.

[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Recalling that the viscous heating rate is $\mathcal{H} = \frac{9}{4}\bar{\nu}\Sigma\Omega^2$, show that the luminosity of the disc, L, may be written as

$$L = \Theta\Omega_{\text{in}} + \frac{1}{2}GM\dot{M}\left(\frac{1}{r_{\text{out}}} - \frac{1}{r_{\text{in}}}\right),$$

where $\Omega_{\text{in}} = \Omega(r_{\text{in}})$. Explain the origins of the various terms and their contributions to the energy budget. Hence determine the energy removed by the wind.

(iv) The spectral energy flux of the disc is given by

$$F_\omega \propto \omega^3 \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{r}{\exp[\epsilon\omega/T(r)] - 1} dr,$$

where ω is the frequency of the radiation, $T(r)$ is the disc surface temperature, and ϵ is a constant.

Supposing $r_{\text{out}} \gg r_{\text{in}}$, explain why $T \approx T_0(r/r_{\text{in}})^{-7/8}$ for radii much smaller than r_{out}, where T_0 is a dimensional constant you need not evaluate.

Adopting this approximation generally, show that $F_\omega \propto \omega^{5/7}$, for intermediate frequencies $T(r_{\text{out}}) \ll \epsilon\omega \ll T(r_{\text{in}})$, where the proportionality constant involves a dimensionless integral you need not evaluate.

2 Disc vertical structure with radiation pressure

The equations determining the equilibrium vertical structure of a thin turbulent Keplerian disc, composed of a perfect gas and radiation, are

$$\frac{dP}{dz} = -\rho\Omega^2 z, \qquad \frac{dF}{dz} = \frac{9}{4}\mu\Omega^2,$$

$$\frac{dT}{dz} = -\frac{3}{16}\frac{\kappa\rho}{\sigma T^3}F, \qquad P = \frac{k\rho T}{\mu_m m_p} + \frac{4\sigma T^4}{3c},$$

where μ is the dynamical turbulent viscosity, κ is opacity, and the other symbols take their usual meanings. Adopt a Thomson opacity model for κ.

(a) Explain the origin of each of the equations and the terms appearing in them.

(b) State the alpha-viscosity prescription for a disc and provide physical arguments for its justification.

(c) Denote the ratio of gas pressure over radiation pressure by β and suppose that it is non-zero and independent of z (though not necessarily a free parameter). Solve the hydrostatic balance equation, showing that

$$T = T_0 f(z), \qquad \rho = \rho_0 f(z)^3, \qquad P = P_0 f(z)^4,$$

where the subscript '0' indicates the midplane value, while $f(z) = 1 - z^2/H^2$, with H the disc semi-thickness. Give expressions for H, ρ_0, and P_0 in terms of T_0, β, and the remaining physical constants of the problem. Show also that

$$H = \frac{1}{A}\left(\frac{ck}{\mu_m m_p \sigma}\right)\frac{\Sigma}{\beta T_0^3},$$

where Σ is surface density and A is a dimensionless integral you need not evaluate.

(d) A vertically averaged alpha parameter may be defined via

$$\int_{-H}^{H} \mu\, dz = \frac{\alpha}{\Omega}\int_{-H}^{H} P\, dz.$$

Denote the total (i.e., vertically integrated) heating rate and cooling rate by Γ and Λ, respectively. Show that, for fixed Σ and α,

$$\Gamma \propto (1+\beta) H T_0^4, \qquad \Lambda \propto \frac{T_0}{\beta H}.$$

Hence find the power-law dependencies of Γ and Λ on T_0 in the limit of (i) a gas-pressure dominated disc and (ii) a radiation-pressure dominated disc.

By drawing a figure, or otherwise, determine the thermal stability of any thermal equilibrium in these two limits.

Noting that β is a decreasing function of T_0, at most how many thermal equilibria are possible in the system (for fixed Σ, α, and Ω)? Justify your answer.

3 Gravitational instability with thermal relaxation

(a) Define the Toomre Q stability parameter for a razor-thin isothermal astrophysical disc. Explain the several physical processes at work in the stability of the disc and how they manifest in the Toomre Q.

Show that the instability criterion can be reworked into $M_D/M_\star \gtrsim H/r$, where M_D and $M_\star$ are the disc's and star's masses, respectively, H is the semi-thickness, and r is cylindrical radius.

(b) The disc's local dynamics can be described by the razor-thin 2D shearing sheet model. Assume the disc is in Keplerian rotation, is now a *perfect gas* with adiabatic index $\gamma\,(> 1)$, and relaxes to a uniform thermal equilibrium on a constant timescale τ. The governing equations are

$$\partial_t \Sigma + \mathbf{u}\cdot\nabla\Sigma = -\Sigma\nabla\cdot\mathbf{u}, \qquad \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -2\Omega\mathbf{e}_z\times\mathbf{u} + 3\Omega^2 x\mathbf{e}_x - \nabla\Phi_{D,m} - \frac{1}{\Sigma}\nabla P,$$

$$\partial_t P + \mathbf{u}\cdot\nabla P = -\gamma P\nabla\cdot\mathbf{u} - \frac{(c^2-c_0^2)\Sigma}{\tau}, \qquad \nabla^2\Phi_D = 4\pi G\Sigma\delta(z), \qquad P = c^2\Sigma,$$

where Σ is surface density, Φ_D is the disc's gravitational potential, $\Phi_{D,m}$ is Φ_D evaluated at $z=0$, c is the (non-constant) isothermal sound speed, c_0 is a constant, δ is the Dirac delta function, and the other symbols take their usual meanings. In the following, assume the disc is axisymmetric.

(i) If $\tilde{}$ denotes the radial Fourier transform, show that $\widetilde{\Phi}_{D,m} = -2\pi G\widetilde{\Sigma}/|k|$, where k is the radial wavenumber.

(ii) Perturb the equilibrium state, $\mathbf{u} = -\frac{3}{2}\Omega x\mathbf{e}_y$, $\Sigma = \Sigma_0$, $c^2 = c_0^2$, by small disturbances and write down the linearised equations governing their evolution. Assume that the perturbations are $\propto \mathrm{e}^{\mathrm{i}kx+st}$, for possibly complex s, and thus derive the dispersion relation

$$s^2 + \Omega^2 - 2\pi G\Sigma_0|k| + c_0^2k^2\left(\frac{\gamma\tau s+1}{\tau s+1}\right) = 0.$$

Interpret the behaviour of the modes in the two limits, $\tau|s| \ll 1$ and $\tau|s| \gg 1$.

(iii) Show that $\mathrm{Im}(s) = 0$ at marginal stability. Calculate the wavenumbers of the marginal modes and hence find an instability criterion.

(iv) Examine the limit $\tau\Omega \equiv \epsilon \ll 1$. By expanding s in powers of ϵ, find the leading-order damping rate of density waves.

END OF PAPER