UNIVERSITY OF CAMBRIDGE

MAMA/317, NST3AS/317, MAAS/317, PCAYM3/317

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 9:00am to 12:00pm

PAPER 317

STRUCTURE AND EVOLUTION OF STARS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe what is meant by a polytrope with polytropic index n. The dimensionless radial variable ξ is such that the radius $r = \alpha\xi$, with $\alpha = \text{const}$, and the dimensionless variable θ is such that the density ρ is related to the central density ρ_{c} by $\rho = \rho_{\mathrm{c}}\theta^n$. Derive the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n,$$

giving α explicitly.

What are the appropriate boundary conditions when the polytrope represents a star? Deduce that the mass inside the sphere of radius r is given by

$$m = -4\pi\rho_{\mathrm{c}}\alpha^3\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}.$$

Show that when $n = 5$ there is a solution to the Lane–Emden equation of the form

$$\theta = (1 + \beta\xi^2)^\gamma,$$

where β and γ are constants which you should identify.

Find the mass, in terms of ρ_{c} and α, and mean density of the star in this case.

Suppose such a polytrope represents the core of a star burning hydrogen by the CNO-cycle in which the central temperature is T_{c}, the equation of state is that of a perfect gas and the energy generation rate per unit mass is given by $\epsilon = \epsilon_0\rho T^{11}$. Deduce that the total luminosity of the burning core is

$$L = A\epsilon_0\alpha^3\rho_{\mathrm{c}}^2T_{\mathrm{c}}^{11}$$

and estimate the order of magnitude of the dimensionless constant A.

Why is this unlikely to be a good model in practice?

[You may find a substitution $x = \sinh\theta$ useful and you may use the fact that $\int_0^1 u^2(1-u^2)^m du = m!(m+1)!2^{2m+1}/(2m+3)!$ when m is an integer.]

2 At a given temperature T the nuclear energy generation rate per unit mass in a stellar interior may be approximated by

$$\epsilon = \epsilon_0 \rho^a T^b,$$

where ρ is the density and ϵ_0 is a constant. Describe briefly the physical origin of the powers a and b and state them at $T \approx 2 \times 10^7\,\mathrm{K}$ for (i) the pp-chain and (ii) the CNO cycle.

A star is in hydrostatic equilibrium and behaves locally as a perfect gas. Show that close to the centre of the star at radius r

$$T(r) = T_{\mathrm{c}}[1 - \alpha r^2 + O(r^4)]$$

and

$$\rho(r) = \rho_{\mathrm{c}}[1 - \beta r^2 + O(r^4)],$$

where T_{c} and ρ_{c} are the central temperature and density and $\alpha > 0$ and $\beta > 0$.

Explain briefly why nuclear burning is typically strongly concentrated toward the centre of the star.

Show that the luminosity generated inside a small radius r behaves as

$$L(r) \propto r^3 \left(1 - \gamma r^2 + \cdots\right)$$

and determine γ in terms of a, b, α, and β.

Let the characteristic burning radius r_{burn} be the radius of the shell of thickness δr in which the nuclear energy generation reaches a maximum. Estimate how r_{burn} scales with the temperature exponent b and the ratio of the sizes of the cores of stars burning hydrogen by the CNO cycle to those dominated by the pp-chain.

Use mass conservation and hydrostatic equilibrium to show that the pressure near the centre of the star obeys

$$P(r) = P_{\mathrm{c}} - \frac{2\pi}{3} G \rho_{\mathrm{c}}^2 r^2 + O(r^4).$$

For a perfect gas with uniform mean molecular weight μ the pressure P is given by

$$P = \frac{\rho \mathcal{R} T}{\mu},$$

where $\mathcal{R}$ is the gas constant. Deduce that

$$\gamma = \frac{3}{5}\left[(2-b)\beta + \frac{2\pi b}{3}\frac{G\rho_{\mathrm{c}}\mu}{\mathcal{R}T_{\mathrm{c}}}\right].$$

3 Explain what is meant by homologous stellar models.

From the equations of stellar structure show that for two homologous stars composed of material of the same chemical composition that obeys a perfect gas law equation of state then their central density, pressure and temperature scale as power laws in the stellar mass M and radius R.

Assume a star on the zero-age main sequence is in radiative equilibrium, with opacity

$$\kappa = \kappa_0 \rho^a T^b,$$

where ρ is the density and T the temperature, with κ_0, a and b constants, and that its nuclear energy generation rate may be approximated by

$$\epsilon = \epsilon_0 \rho T^n,$$

where ϵ_0 and n are constants. Derive how the luminosity L scales with mass M in terms of n, a and b.

What does this mean for the mass–luminosity relation and the lifetime of massive stars?

Consider a sequence of stars of increasing mass for which radiation pressure becomes non-negligible. Why must the slope of the mass–luminosity relation eventually flatten and what is the upper limit in terms of the mass of the star and its opacity? What does this mean for stellar lifetimes?

A star evolves quasi-statically while remaining approximately homologous during a phase of core hydrogen burning. Explain why the mean molecular weight in the core increases with time and indicate qualitatively how the star's radius and luminosity evolve.

Explain why homology eventually fails as the star leaves the main sequence.

4 A binary system consists of a star of mass M_1 and radius R_1 orbiting a companion of mass M_2 in a circular orbit of separation a. Show that the orbital angular momentum of the binary may be written as

$$J = \mu\sqrt{GMa},$$

where $M = M_1 + M_2$ and $\mu = M_1M_2/M$ is the reduced mass and G is Newton's gravitational constant.

Hence show that

$$\frac{\dot{a}}{a} = 2\frac{\dot{J}_{\rm orb}}{J_{\rm orb}} - 2\frac{\dot{M}_1}{M_1} - 2\frac{\dot{M}_2}{M_2} + \frac{\dot{M}}{M}.$$

For conservative mass transfer from star 1 to star 2, neglecting the spin angular momentum of the stars, show that the orbital separation evolves according to

$$\frac{\dot{a}}{a} = -2\dot{M}_1\left(\frac{1}{M_1} - \frac{1}{M_2}\right).$$

Hence show that conservative mass transfer causes the orbit to widen when $M_1 < M_2$, but shrink when $M_1 > M_2$.

The Roche-lobe radius of star 1 may be approximated by

$$\frac{R_{\rm L}}{a} = 0.49\,\frac{q^{2/3}}{0.6q^{2/3} + \ln(1+q^{1/3})}, \qquad q = \frac{M_1}{M_2}.$$

Show that, for small fractional changes,

$$\frac{d\ln R_{\rm L}}{d\ln M_1} = \frac{d\ln a}{d\ln M_1} + (1+q)f(q),$$

giving $f(q)$ explicitly.

Suppose that on timescales short compared to its evolutionary timescale the donor star responds to mass loss according to

$$R_1 \propto M_1^{-n}, \qquad n > 0.$$

Show that mass transfer is dynamically stable if

$$n < n_{\rm crit}(q),$$

and find $n_{\rm crit}(q)$ in terms of q and $f(q)$.

Argue that mass transfer is always dynamically unstable when the donor star is more massive and explain briefly what physical outcome is expected in that case.

END OF PAPER