UNIVERSITY OF CAMBRIDGE

MAMA/316, NST3AS/316, MAAS/316, PCAYM3/316

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 4 June 2026 2:00pm to 5:00pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) A planetesimal of mass m is in orbit around a star of mass $M_\star$. The equation of relative motion is $\ddot{\boldsymbol{r}} + \mu\boldsymbol{r}/r^3 = 0$, where $\mu = G(M_\star + m)$, $\boldsymbol{r}$ is the vector offset of the planetesimal from the star, $r = |\boldsymbol{r}|$, G is the gravitational constant, and dots denote differentiation with respect to time. Show that there are two constants of motion $h = r^2\dot{\theta}$ and $C = (1/2)v^2 - \mu/r$, where θ is the azimuthal angle in the planet's orbital plane relative to a reference direction, and v is the velocity of the planetesimal relative to the star.

(b) Derive the solution to the equation of relative motion given above to find that $r = a(1-e^2)/(1+e\cos f)$, where the orbital elements a, e and f have their usual meaning, and determine the constants h and C in terms of those orbital elements.

(c) Consider the reference frame centred on the star with the unit vector $\hat{\boldsymbol{x}}$ pointing towards the pericentre of the orbit and the unit vector $\hat{\boldsymbol{y}}$ orthogonal to that in the orbital plane in the direction of motion. Derive expressions for the velocity of the planetesimal relative to the star in the x and y directions, v_x and v_y.

(d) When the planetesimal is at a true anomaly f it releases a particle at a velocity γ times the Keplerian velocity of a body on a circular orbit at distance a from the star. The release is in the planetesimal's orbital plane in a direction oriented by an angle ϕ from that of its motion at pericentre. The planetesimal's gravity can be ignored. Show that the semimajor axis of the particle's orbit a' is given by

$$a/a' = 1 - \gamma^2 - 2\gamma(1-e^2)^{-1/2}[\cos(\phi - f) + e\cos\phi].$$

(e) Derive an expression for the eccentricity of the particle's orbit e' as a function of e, f, γ and ϕ.

(f) When the planetesimal reaches pericentre it releases many particles in random directions ϕ, all at the same velocity defined by γ. If one sixth of these particles are on unbound orbits, show that for an orbit with $e = 1/2$, the velocity of release $\gamma = (\sqrt{13} - 3)/2$.

2 (a) Consider a planetary system comprising a star and a planet p. A particle orbiting in the system has a Tisserand parameter with respect to the planet of

$$T_{\mathrm{p}} = (a_{\mathrm{p}}/a) + 2\sqrt{(a/a_{\mathrm{p}})(1-e^2)}\cos I.$$

Describe the meaning of the different parameters, the assumptions made in the derivation of the above relation, the context under which the Tisserand parameter is conserved, and the physical interpretation of the two terms on the right hand side.

(b) The particle undergoes multiple scattering events with the planet. Consider the locus of possible particle orbits on a standard plot of its eccentricity versus semimajor axis for fixed inclination relative to the planet. Find the locations of any maxima or minima on such a plot (which does not need to be provided at this stage), including any constraints on T_{p} for these to exist.

(c) Sketch the locus described in (b) for a particle with $T_{\mathrm{p}} = 5/2$ that starts off coplanar with the planet.

(d) What inclination relative to the planet's orbit would a particle with $T_{\mathrm{p}} = 5/2$ require to have the possibility of achieving a circular orbit?

(e) Describe the phase space of possible orbits (in eccentricity versus semimajor axis) for a particle with $T_{\mathrm{p}} = 5/2$ that is started off non-coplanar with the planet.

(f) Without detailed calculation, describe how T_{p} determines how far the particle is likely to move on the eccentricity versus semimajor axis plot in each encounter with the planet.

3 (a) A star hosts a debris belt at radial distance r from the star. The size distribution of bodies in the belt is such that the number with diameters in the range D to $D + dD$ is $n(D)dD$, where $n(D) \propto D^{-\alpha}$ and $3 < \alpha < 4$. These bodies may be assumed to be spherical and to act as black bodies, with diameters extending from $D_{\rm min}$ to $D_{\rm max} \gg D_{\rm min}$. If the fractional luminosity from the belt is f, derive an expression for the number of bodies in the belt that are larger than D, i.e., $n(> D)$.

(b) Collisions within the belt occur at relative velocity $v_{\rm rel}$ and the bodies are uniformly distributed within the belt's volume V. If the bodies have a dispersal threshold $Q^\star_{\rm D}$, and gravitational focussing can be ignored, show that the rate of catastrophic collisions for bodies of diameter D is given by

$$R_{\rm cc}(D) \approx A v_{\rm rel}^{(1+2\alpha)/3} (Q^\star_{\rm D})^{(1-\alpha)/3} D^{3-\alpha},$$

where the constant A should be determined, and any further assumptions needed to get this result should be specified.

(c) Hence give an approximate expression for the mean time between collisions in the belt involving the destruction of objects larger than a size D, i.e., $t_{\rm m}(> D)$.

(d) The dispersal threshold is size dependent, such that $Q^\star_{\rm D} \propto D^b$, where b is a constant. If the size distribution has reached steady state, show that $\alpha = (21 + b)/(6 + b)$, explaining all of your reasoning.

(e) Show that the mean time between collisions in such a steady state distribution is $t_{\rm m}(> D) \propto D^3$, and comment on the dependence on the dispersal threshold index b.

(f) A commonly used dispersal threshold is $Q^\star_{\rm D} = Q_a D^{-a} + Q_b D^{-b}$, where Q_a, Q_b, a and b are positive constants. A rubblising collision is one with sufficient energy to shatter the body catastrophically but that is unable to disperse the fragments due to their self gravity. Estimate the number of rubblising collisions that a planetesimal in the gravity regime experiences before it is finally catastrophically destroyed.

4 A dust particle with constant radiation pressure coefficient β is spiralling in towards its host star of mass $M_\star$ due to Poynting-Robertson (P-R) drag. Its orbit has a small but non-zero eccentricity $e_{\rm d}$ and longitude of pericentre $\varpi_{\rm d}$ when it encounters the 5:4 mean motion resonance with a planet of mass $M_{\rm p} \ll M_\star$ which is on a circular coplanar orbit at a distance $a_{\rm p}$ from the star. The mean longitudes of the dust particle and planet are $\lambda_{\rm d}$ and $\lambda_{\rm p}$, respectively.

(a) Give the (approximate) semimajor axis of the particle $a_{\rm d}$, and identify the dominant term in the disturbing function from perturbations to its orbit due to the planet, commenting on both the strength of the term and its argument ϕ.

(b) Explaining your reasoning, plot the orbit of the dust particle in a frame that is centred on the star and rotating with the planet. The plot should be drawn for conjunctions that occur at the apocentre of the particle's orbit for $e_{\rm d} = 0.2$.

(c) Describe how you expect perturbations from the planet's gravity to alter the particle's angular momentum, assuming that such perturbations are dominated by those at conjunction, including how those expectations depend on the longitude of conjunction relative to the particle's apocentre.

(d) P-R drag causes the particle's orbit to lose angular momentum, which can be counteracted by planetary perturbations causing the particle to become trapped in resonance. How do you expect the pattern to be oriented on the plot made in (b) for such resonantly trapped dust, and what does this mean for the angle ϕ?

(e) Taking terms to first order in eccentricity, planetary perturbations and P-R drag cause the particle's orbit to evolve as

$$\begin{aligned} \dot{a}_{\rm d} &= -8 e_{\rm d} a_{\rm d} D \sin\phi - 2(\beta G M_\star / c)/a_{\rm d}, \\ \dot{e}_{\rm d} &= -D \sin\phi - (5/2)(\beta G M_\star / c) e_{\rm d}/a_{\rm d}^2, \end{aligned}$$

where D is a positive constant, G is the gravitational constant, c is the speed of light. Determine a constraint on β for resonant trapping to be possible.

(f) Show that the evolution of the particle's eccentricity while in resonance is given by $e_{\rm d} = \sqrt{e_{\rm d0}^2 + 2De_{\rm d0}(\beta/\beta_{\rm max})t}$, where $e_{\rm d0}$ and $\beta_{\rm max}$ should be defined.

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