UNIVERSITY OF CAMBRIDGE

MAMA/315, NST3AS/315, MAAS/315, NST3QC/ESP, MAAMM2/ESP, PCAYM3/315

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 11 June 2026 2:00pm to 5:00pm

PAPER 315

EXTRASOLAR PLANETS: ATMOSPHERES AND INTERIORS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the temperature profile of an irradiated planetary atmosphere in radiative equilibrium expressed in the following form

$$T^4(\tau) = aT_{\rm int}^4\,(b+\tau) + afT_{\rm irr}^4\left(b + \frac{1}{\gamma c} + \left(\frac{\gamma}{c} - \frac{1}{\gamma c}\right)e^{-\gamma c\tau}\right), \qquad \text{(I)}$$

where a, b, and c are constants and f can be assumed to be 1/2.

a. What do the quantities $T_{\rm int}$, $T_{\rm irr}$, γ and τ represent? How are a and b related? Discuss the possible atmospheric P-T profiles for three types of planets: (1) a giant planet in a 40 au orbit around a young star, (2) a highly irradiated hot jupiter orbiting a sun-like star, and (3) a temperate sub-Neptune orbiting an M dwarf star. Sketch the P-T profile in each case.

b. Consider the deep atmosphere of a hot jupiter where thermochemical equilibrium is typically satisfied. Estimate the range of τ in this region where the temperature profile is expected to be isothermal and estimate the corresponding temperature. How does the temperature in the radiative region vary below this region? Make and state any assumptions needed.

c. Derive the pressure-temperature gradient (dT/dP) from the profile shown in (I).

d. Estimate the pressure at the radiative-convective boundary for the atmosphere considered in part b.

e. Derive the condition for a thermal inversion in the observable atmosphere of a highly irradiated planet. Explain what factors typically affect this condition for three types of planets (1) Jupiter in the solar system, (2) an ultra hot jupiter around a 6500 K star, and (3) a temperate sub-Neptune planet orbiting a 2500 K star.

2

a. (i) A Neptune-sized planet (CAMX-2 b) orbiting a sun-like star has a uniform geometric albedo $A_{\rm g}$ below 1 μm and an isothermal atmosphere with a Bond albedo of 0.5. Derive the expression for the planet-star flux ratio ($F_{\rm p}/F_{\rm s}$) observed at secondary eclipse. Assuming the planet has an equilibrium temperature $T_{\rm eq}$ of 600 K, sketch the planet-star flux ratio from the visible to mid-infrared wavelength range.

(ii) A space telescope has two photometric bands, one centred at 0.5 μm (Band 1) and another at 20 μm (Band 2). This telescope was used to measure a median value of $F_{\rm p}/F_{\rm s}$ of CAMX-2 b in Band 1 to be 50 ppm. Estimate $A_{\rm g}$ for this planet and make a prediction for the expected $F_{\rm p}/F_{\rm s}$ in Band 2.

(iii) Consider a perfectly diffusing Lambertian disk of radius R with an incident flux F. Derive an expression for the flux from the disk as seen at a distance d.

b. Consider a one-dimensional planetary atmosphere in local thermodynamic equilibrium with the following temperature profile and no scattering.

$$T(\tau) = \begin{cases} T_1 & \text{for } 0 < P < P_1 \\ T_2 & \text{for } P_1 < P < P_2 \\ T_3 & \text{for } P > P_2 \end{cases},$$

where T_1, T_2, and T_3 are constants.

(i) Consider a ray of specific intensity $I_{\nu,0}$ at the bottom of the atmosphere travelling outward. Derive an expression for the emergent intensity at the top of the atmosphere. Derive an expression for the planet-star flux ratio in Band 2 defined above assuming the same planetary and stellar sizes as the CAMX-2 b system. Evaluate this expression for $T_1 = T_2 = T_3 = T_{\rm p}$.

(ii) Suppose that $T_1 = 400$ K, $T_2 = 600$ K and $T_3 = 800$ K, and that the average optical depth in Band 2 defined above is 0.1 and 1.0 at pressures P_1 and P_2, respectively. Estimate the planet-star flux ratio in Band 2 and compare it with that for the isothermal case in part a(ii) above for the same planetary and stellar sizes.

3

a. The transmission spectrum of a Jupiter-mass exoplanet orbiting a sun-like star was observed to be of the form

$$y = a + be^{-\alpha(\lambda-\lambda_0)^2} + c(\lambda/\lambda_1)^\gamma, \tag{1}$$

where y is the transit depth, λ is the wavelength, α, γ, λ_0 and λ_1 are constants and a, b, and c are quantities related to the planetary properties.

(i) Discuss what could be causing each of the three components in this expression.

(ii) Derive the planetary radius and the average temperature of the day-night terminator region of the atmosphere in terms of the quantities shown and any other constants required.

(iii) Considering a realistic atmosphere of a hot jupiter, estimate the values for a and b and sketch the transmission spectrum showing all three components. Make and state any assumptions needed.

b. A hot Neptune transiting an M dwarf star half the size of the sun was observed in the ultraviolet using the Hubble Space Telescope. The observations revealed an asymmetric light curve with a transit depth of 0.5.

(i) What could be causing the asymmetry and the large transit depth?

(ii) Derive an approximate condition under which hydrostatic thermal escape can cause atomic species to escape a planet. Derive an expression for the temperature at the base of the exosphere required for escape of atomic hydrogen from this planet.

(iii) Considering the escape parameter, λ, discuss the range of values for which hydrostatic escape dominates over hydrodynamic escape and vice versa. Derive an expression for the mass loss rate if the escape parameter was found to be $\lambda \leqslant 1$.

(iv) State three non-thermal mechanisms which can cause atmospheric escape in planetary atmospheres.

c. Assume that you have access to the James Webb Space Telescope (JWST) and your goal is to detect the spectral signature of a molecule in the atmosphere of a potentially habitable exoplanet orbiting a nearby bright star.

(i) State three kinds of habitable exoplanets that might be accessible to JWST with examples of known planets. What is the challenge in making such a detection for an Earth-twin?

(ii) What factors affect the observability of the atmosphere in transmission and emission spectroscopy?

(iii) Assuming typical parameters for each of them, estimate the amplitude of a spectral feature in a transmission spectrum for each planet type discussed above. Which of these planet types is most observable with JWST? Make and state any assumptions needed.

4

a. Discuss how the day-night temperature contrast observed for a hot jupiter atmosphere depends on the incident irradiation and the pressure-level in the atmosphere.

b. Starting from the radiative transfer equation demonstrate how the temperature gradient of an atmosphere affects the observed spectral features of a planet.

c. Discuss three atmospheric processes that typically affect the chemical composition in different regions of an irradiated atmosphere, providing an example for each. Consider a temperate sub-Neptune exoplanet with Earth-like equilibrium temperature. How does the presence or absence of a surface affect the chemical composition of the planet? How does the expected composition differ from that of a hot jupiter?

d. Consider an exoplanet A with a mass of 10 Earth masses and a radius of 2 Earth radii. The planet has an iron core and a silicate mantle, each of 4 Earth masses, with the rest of the mass in H_2. Sketch a ternary diagram and mark the location of this planet on that diagram. Mark the location of the Earth on the same diagram.

e. Show that the specific intensity of radiation received from a source is invariant with distance from the source. Derive a relation involving the mean intensity, surface flux and/or the K-integral when the specific intensity varies linearly with the direction cosine.

END OF PAPER