UNIVERSITY OF CAMBRIDGE

MAMA/314, NST3AS/314, MAAS/314, PCAYM3/314

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2:00pm to 5:00pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho &= -\rho \boldsymbol{\nabla} \cdot \mathbf{u} \,, \\
\frac{\partial p}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} p &= -\gamma p \boldsymbol{\nabla} \cdot \mathbf{u} \,, \\
\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \right) &= -\rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} p + \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} \,, \\
\frac{\partial \mathbf{B}}{\partial t} &= \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \,, \\
\nabla^2 \Phi &= 4\pi G \rho \,.
\end{aligned}$$

You may assume that, for vector fields $\mathbf{C}$ and $\mathbf{D}$,

$$\begin{aligned}
(\boldsymbol{\nabla} \times \mathbf{C}) \times \mathbf{C} &= \mathbf{C} \cdot \boldsymbol{\nabla} \mathbf{C} - \boldsymbol{\nabla} \left(\frac{1}{2} |\mathbf{C}|^2 \right) , \\
\boldsymbol{\nabla} \cdot (\mathbf{C} \times \mathbf{D}) &= \mathbf{D} \cdot (\boldsymbol{\nabla} \times \mathbf{C}) - \mathbf{C} \cdot (\boldsymbol{\nabla} \times \mathbf{D}) \,, \\
\boldsymbol{\nabla} \times (\mathbf{C} \times \mathbf{D}) &= \mathbf{D} \cdot \boldsymbol{\nabla} \mathbf{C} - \mathbf{C} \cdot \boldsymbol{\nabla} \mathbf{D} - \mathbf{D}(\boldsymbol{\nabla} \cdot \mathbf{C}) + \mathbf{C}(\boldsymbol{\nabla} \cdot \mathbf{D}) \,.
\end{aligned}$$

1 (a) A star is modelled as a perfect gas, of adiabatic exponent γ, in the gravitational potential $\Phi = \frac{1}{2}\Omega^2|\mathbf{x}|^2$, where Ω is a positive constant. In this question the potential Φ is regarded as fixed and the self-gravity of the star is neglected.

Consider fluid motions of the form

$$\mathbf{u} = (Ax, By, Cz)\,,$$

where $A(t)$, $B(t)$ and $C(t)$ are functions to be determined. Assume that the star occupies the time-dependent ellipsoidal volume $f > 0$ with a free surface at $f = 0$, where

$$f = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right),$$

with $a(t)$, $b(t)$ and $c(t)$ being positive functions. How must A, B and C be related to a, b and c in order for the free surface to move with the fluid? What does this imply about the Lagrangian derivative of f inside the ellipsoid?

(b) Assume further that the density and pressure have the form

$$\rho = \rho_0(t)\widehat{\rho}(f)\,, \qquad p = \rho_0(t)T(t)\widehat{p}(f)\,,$$

where $\rho_0(t)$ and $T(t)$ are positive functions and $\widehat{\rho}(f)$ and $\widehat{p}(f)$ are positive dimensionless functions related by $d\widehat{p}/df = \widehat{\rho}$ and satisfying the boundary condition $\widehat{p}(0) = 0$. Show that the equations of gas dynamics and the boundary conditions on the free surface are satisfied, provided that the coefficients evolve according to

$$\begin{gathered} \ddot{a} + \Omega^2 a = \frac{2T}{a}\,, \qquad \ddot{b} + \Omega^2 b = \frac{2T}{b}\,, \qquad \ddot{c} + \Omega^2 c = \frac{2T}{c}\,, \\ \rho_0 \propto (abc)^{-1}\,, \qquad T \propto (abc)^{-(\gamma-1)}\,. \end{gathered} \tag{$\star$}$$

(c) Find a conserved quantity for the system ($\star$) that is proportional to the total energy of the star.

(d) Linearize the system ($\star$) about a spherical hydrostatic equilibrium state in which $a = b = c$, to show that

$$\frac{\ddot{\delta a}}{a} = \Omega^2\left(-2\,\frac{\delta a}{a} + \frac{\delta T}{T}\right),$$

and similarly for b and c. Hence show that the angular frequencies ω of the oscillation modes are given by $\omega/\Omega = \sqrt{3\gamma - 1}$ or $\omega/\Omega = \sqrt{2}$ and comment on their physical interpretation.

(e) When the star has a binary companion with an eccentric orbit, you may assume that the star experiences (in addition to the fixed potential Φ) a tidal potential component of the form

$$\Psi = \Psi_0(2z^2 - x^2 - y^2)\cos(\Omega_{\mathrm{o}} t)\,,$$

where Ψ_0 is a constant (proportional to the eccentricity of the orbit) and Ω_{o} is the mean angular velocity of the orbit. Determine how the two types of oscillation mode in part (d) respond to this type of tidal forcing.

2 (a) Consider a plane-parallel atmosphere with uniform gravity $\mathbf{g} = -g\,\mathbf{e}_z$ (where g is a positive constant). Show that the hydrostatic structure of a polytropic atmosphere with a free surface at $z = 0$, in which the equilibrium pressure and density are related by $p \propto \rho^{1+1/m}$, where m is a positive constant, is given by

$$\rho = \rho_0 \left(\frac{-z}{H}\right)^m , \qquad z \leqslant 0 \,,$$

where ρ_0 and H are positive constants. Find the corresponding expression for the equilibrium pressure.

(b) The atmosphere is neutrally stratified such that $1 + 1/m = \gamma$, the adiabatic exponent of the gas. It also rotates uniformly around the z-axis with angular velocity Ω.

Consider small perturbations of the atmosphere, in which the density perturbation, in a frame of reference rotating with the atmosphere, has the form

$$\mathrm{Re}\left[\delta\rho(z)\exp(ik_x x - i\omega t)\right] ,$$

and similarly for the velocity and pressure perturbations, where $k_x > 0$ is the horizontal wavenumber and $\omega \neq 0$ is the angular frequency. Show that the linearized equations are

$$\begin{aligned}
-i\omega\rho\,\delta u_x - 2\Omega\rho\,\delta u_y &= -ik_x\,\delta p \,, \\
-i\omega\rho\,\delta u_y + 2\Omega\rho\,\delta u_x &= 0 \,, \\
-i\omega\rho\,\delta u_z &= -g\,\delta\rho - \frac{d\,\delta p}{dz} \,, \\
-i\omega\,\delta\rho + \delta u_z\,\frac{d\rho}{dz} &= -\rho\left(ik_x\,\delta u_x + \frac{d\,\delta u_z}{dz}\right) , \\
-i\omega\,\delta p + \delta u_z\,\frac{dp}{dz} &= -\gamma p\left(ik_x\,\delta u_x + \frac{d\,\delta u_z}{dz}\right) .
\end{aligned}$$

[You may assume that self-gravity is negligible, and that the centrifugal force is either negligible or has been combined with the gravitational force.]

(c) Rewrite the linearized equations in terms of the displacement $\boldsymbol{\xi}$. Show that $\delta p/p = \gamma\,\delta\rho/\rho$ and so reduce the equations to the form

$$\begin{aligned}
-(\omega^2 - 4\Omega^2)\xi_x &= -ik_x\Psi \,, \\
-\omega^2\xi_z &= -\frac{d\Psi}{dz} \,,
\end{aligned}$$

with

$$\Psi = \frac{\delta p}{\rho} = g\xi_z - v_{\mathrm{s}}^2\Delta \,, \qquad \Delta = ik_x\xi_x + \frac{d\xi_z}{dz} \,, \qquad v_{\mathrm{s}}^2 = \frac{\gamma p}{\rho} \,.$$

Hence obtain a pair of coupled ordinary differential equations (ODEs) for $\xi_z(z)$ and $\Psi(z)$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(d) Show that incompressible solutions exist, with

$$\omega^2 = 2\Omega^2 + \left[(gk_x)^2 + (2\Omega^2)^2\right]^{1/2} .$$

(e) Assuming that $\omega > 2\Omega$, write

$$\Psi(z) = \psi(z)\, e^{kz} , \qquad k = \frac{k_x \omega}{(\omega^2 - 4\Omega^2)^{1/2}} ,$$

to obtain the ODE

$$v_{\mathrm{s}}^2 \left(\frac{d^2\psi}{dz^2} + 2k\frac{d\psi}{dz} \right) - g\left(\frac{d\psi}{dz} + k\psi \right) + \omega^2 \psi = 0 .$$

Show that the condition for this ODE to have a polynomial solution of degree n is

$$\omega^2 = \left(\frac{2n}{m} + 1 \right) gk .$$

Give a physical interpretation of these solutions, including the case $n = 0$. [*Hint*: Consider the highest power of z in the equation. You are not required to obtain a recurrence relation for the polynomial coefficients.]

3 To analyse shocks in magnetohydrodynamics, consider a frame of reference in which the shock is stationary and in the plane $x = 0$, with the gas flowing from region 1 ($x < 0$) to region 2 ($x > 0$). Assume that the velocity and magnetic field lie in the xy-plane and have strictly positive x-components. Assume that the gas is perfect, with adiabatic exponent γ.

(a) Justify the jump conditions

$$\begin{aligned}
[\rho u_x] &= 0, \\
\left[\rho u_x^2 + \Pi - \frac{B_x^2}{\mu_0}\right] &= 0\,, \\
\left[\rho u_x u_y - \frac{B_x B_y}{\mu_0}\right] &= 0\,, \\
[B_x] &= 0\,, \\
[-E_z] = [u_x B_y - u_y B_x] &= 0\,, \\
\left[\rho u_x\left(\frac{1}{2}u^2 + h\right) - \frac{E_z B_y}{\mu_0}\right] &= 0\,,
\end{aligned}$$

where Π is the total pressure, h is the specific enthalpy and $[X] = X_2 - X_1$ denotes the jump in any quantity from region 1 to region 2. Why does gravity not appear in the jump conditions? Why can the entropy fluxes in regions 1 and 2 not be assumed to be equal, if dissipation is allowed to occur within the shock? [You may quote alternative forms of the MHD equations from those given in the rubric if these are helpful in your argument.]

(b) Explain why the problem can be transformed to a frame in which the electric field vanishes and $\mathbf{u} \parallel \mathbf{B}$. Simplify the jump conditions using this frame.

(c) Show that (when these ratios are not indeterminate)

$$\frac{\rho_2}{\rho_1} = \frac{\mathcal{A}_1^2}{\mathcal{A}_2^2}\,, \qquad \frac{B_{y2}}{B_{y1}} = \frac{\mathcal{A}_1^2 - 1}{\mathcal{A}_2^2 - 1}\,,$$

where $\mathcal{A}_i$ is the ratio of the velocity to the Alfvén velocity in region i and B_{yi} is the value of B_y in region i.

(d) Suppose that the downstream magnetic field has no component tangential to the shock. Show that:

(i) if the upstream magnetic field also has no tangential component, then the jump conditions reduce to the Rankine–Hugoniot relations of gas dynamics (which you are not required to solve);

(ii) if the upstream magnetic field has a non-zero tangential component, then the upstream velocity and Alfvén velocity are equal.

(e) Show that the jump conditions can also be satisfied if the tangential components of the magnetic field are equal and opposite in regions 1 and 2, with the velocity equalling the Alfvén velocity in both regions. How do the density and pressure behave across the interface in this case?

4 Consider a steady, axisymmetric outflow from a magnetized accretion disc around a star of mass M. The gravity of the disc can be neglected, so the gravitational potential is $\Phi = -GM(r^2+z^2)^{-1/2}$ in cylindrical polar coordinates (r,ϕ,z). You may assume (i) that the magnetic field has the poloidal–toroidal decomposition

$$\mathbf{B} = \mathbf{B}_\mathrm{p} + \mathbf{B}_\mathrm{t} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi + rB_\phi\,\boldsymbol{\nabla}\phi\,,$$

where ψ and B_ϕ are functions of r and z, (ii) that the velocity field has the form

$$\mathbf{u} = \frac{k\mathbf{B}}{\rho} + r^2\omega\,\boldsymbol{\nabla}\phi\,,$$

where $\rho(r,z)$ is the density, and (iii) that k, ω and $\ell = ru_\phi - \dfrac{rB_\phi}{\mu_0 k}$ are functions of ψ.

(a) Give a physical interpretation of the quantity $\psi(r,z)$.

(b) State (without derivations) which of the MHD equations are solved by the above description and which remain to be solved.

(c) Explain why the (modified) Bernoulli function

$$\tilde{\varepsilon} = \frac{1}{2}u_\mathrm{p}^2 + \frac{1}{2}(u_\phi - r\omega)^2 + \Phi_\mathrm{cg} + h\,, \qquad \text{with} \quad \Phi_\mathrm{cg} = \Phi - \frac{1}{2}\omega^2 r^2\,,$$

is also a function of ψ, despite the Lorentz force. Give a physical interpretation of Φ_cg.

(d) Assuming that the outflow passes smoothly through an Alfvén surface at $r = r_\mathrm{a}(\psi)$, show that

$$B_\phi = \frac{\mu_0 k\omega}{r}\left(\frac{r^2 - r_\mathrm{a}^2}{1-\mathcal{A}^2}\right)\,,$$

where $\mathcal{A}$ is the poloidal Alfvén number (which you should define). How is $\mathcal{A}$ related to ρ along each field line?

(e) Why is it reasonable to identify $\omega(\psi)$ with the angular velocity $\Omega_0 = \sqrt{GM/r_0^3}$ of a circular orbit at the footpoint radius $r_0(\psi)$ of each field line? Making this assumption, show that the expansion of Φ_cg to second order in the neighbourhood of the footpoint $(r,z) = (r_0,0)$ is

$$\Phi_\mathrm{cg} \approx \text{constant} + \frac{1}{2}\Omega_0^2\left[z^2 - 3(r-r_0)^2\right]\,.$$

Hence show that Φ_cg decreases away from the disc along the field line if it is inclined at more than 30° to the vertical direction.

(f) A typical argument for magnetocentrifugal acceleration is that, if both $\frac{1}{2}(u_\phi - r\omega)^2$ and h are negligible in the region of interest, then $\frac{1}{2}u_\mathrm{p}^2$ must increase along a field line if Φ_cg decreases. To investigate the effect of the $\frac{1}{2}(u_\phi - r\omega)^2$ term in the Bernoulli equation, show that the ratio $\mathcal{R} = |u_\phi - r\omega|/u_\mathrm{p}$ equals $|B_\phi|/B_\mathrm{p}$ and can be estimated as

$$\mathcal{R} \approx \tilde{\mathcal{R}} = \frac{r_\mathrm{a}^2\Omega_0\mu_0 k}{r_0 B_\mathrm{p}}$$

in the launching region, where $r^2 \approx r_0^2 \ll r_\mathrm{a}^2$ and $\mathcal{A} \ll 1$. What effect does the $\frac{1}{2}(u_\phi - r\omega)^2$ term in the Bernoulli equation then have on the argument in part (e)?

END OF PAPER