UNIVERSITY OF CAMBRIDGE

MAMA/313, NST3AS/313, MAAS/313

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 16 June 2026 9:00am to 11:00am

PAPER 313

SOLITONS, INSTANTONS AND GEOMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let (A, ϕ) be an abelian gauge potential and a complex scalar field on a surface Σ with a Riemannian metric

$$g = \Omega dz d\bar{z}, \quad \Omega = \Omega(z, \bar{z}).$$

(a) Assume that $\Sigma = \mathbb{R}^2$ and g is flat with $\Omega = 1$.

Define the Landau–Ginsburg energy functional, and show that the Derrick scalling argument allows the existence of solitons for this functional.

Define the vortex number N and show how it is related to the $\int_{\mathbb{R}^2} F$ where $F = dA$.

(b) Going back to general Ω, the Bogomolny equations are

$$D_{\bar{z}}\phi = 0, \quad B = \frac{1}{2}\Omega(1 - |\phi|^2)$$

where $D_{\bar{z}} \equiv \frac{1}{2}(D_x + iD_y)$ and $F = B dx \wedge dy$ with $z = x + iy$.

(i) Derive the Taubes equation for a scalar function u on Σ such that $e^u = |\phi|^2$.

(ii) Assume that $\Omega = 1$. State the boundary conditions satisfied by u if the Higgs field vanishes at one point to order N in $\mathbb{R}^2$ and show that $|\phi| \leqslant 1$.

(iii) Assume that Σ is a compact surface with a metric

$$g = \frac{a^2}{(1 + |z|^2)^2} dz d\bar{z}$$

where a is a constant. Find a condition on a which is neccessary for the existence of a 2–vortex solution.

2 Let A be an $\mathfrak{su}(2)$ valued one–form on $\mathbb{R}^D$.

(a) Define the 2nd Chern form C_2, and show that $C_2 = dY$, where

$$Y = \frac{1}{8\pi^2}\mathrm{Tr}(A \wedge dA + \alpha A \wedge A \wedge A)$$

where α is a constant which should be determined.

(b) Show that if $g : \mathbb{R}^D \to SU(2)$ and

$$A \to gAg^{-1} - dg \cdot g^{-1}$$

then

$$C_2 \to C_2, \quad Y \to Y + dT + \frac{1}{24\pi^2}\mathrm{Tr}([dg \cdot g^{-1}]^3)$$

where T is a two–form, which should be determined.

(c) State the boundary conditions satisfied by a $SU(2)$ Yang–Mills instanton on $\mathbb{R}^4$, and express the instanton number by a topological degree of a gauge transformation at infinity.

3 Let G be a matrix Lie group with a Lie algebra $\mathfrak{g}$ with generators $T_\alpha, \alpha = 1, \ldots, n$ which satisfy

$$[T_\alpha, T_\beta] = \sum_\gamma c^\gamma{}_{\alpha\beta} T_\gamma.$$

(a) Define the Maurer–Cartan one–form ρ on G and show that

$$d\rho + \rho \wedge \rho = 0.$$

(b) Set $\rho = \sum_\alpha \sigma^\alpha T_\alpha$ and show that

$$d\sigma^\gamma = \sum_{\alpha,\beta} f^\gamma_{\alpha\beta} \sigma^\alpha \wedge \sigma^\beta$$

for some constants $f^\gamma{}_{\alpha\beta}$ which should be determined.

(c) Now assume that G is a two–dimensional Lie group acting on $\mathbb{R}$ by $x \to e^a x + b$ where $(a, b) \in \mathbb{R}^2$. Find a matrix representation of G and construct the left–invariant one forms σ^1, σ^2 and the left–invariant vector fields on G.

(d) Consider a Riemannian metric on G given by

$$\mathbf{g} = (\sigma^1)^2 + (\sigma^2)^2,$$

and demonstrate that the right–invariant vector fields on G generate one–parameter groups of isometries of $\mathbf{g}$. Does $\mathbf{g}$ admit a Killing vector which is not right–invariant? [you are not required to find one, but you should give a *yes* or *no* answer with a justification].

END OF PAPER