MAT3

MATHEMATICAL TRIPOS Part III

Thursday 11 June 2026 9:00am to 12:00pm

PAPER 312

FIELD THEORY IN COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the scalar-field theory

$$S[\phi] = \int d^4x \sqrt{-g}\, P(X,\phi), \qquad X \equiv -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\,, \tag{†}$$

where we assume a spatially flat de Sitter background in conformal time τ described by

$$ds^2 = a(\tau)^2(-d\tau^2 + d\mathbf{x}^2)\,, \qquad a(\tau) = -\frac{1}{H\tau}\,, \qquad \tau < 0\,.$$

(i) Expand the field about a homogeneous background as $\phi(\tau,\mathbf{x}) = \bar{\phi}(\tau) + \varphi(\tau,\mathbf{x})$ and, by varying the action (†) at quadratic order, find the equation of motion for the perturbation φ. Up to leading order in a slow-variation approximation, show that the Fourier modes of the perturbation satisfy

$$u_k'' + \left(c_s^2 k^2 - \frac{2}{\tau^2}\right) u_k = 0, \qquad u_k \equiv a\,\varphi_k, \tag{$*$}$$

where you should define the sound speed c_s and clearly specify the assumptions made.

Find linearly independent solutions $u_k(\tau)$ of $(*)$ and, by imposing an appropriate asymptotic initial condition, show that the approximate inflationary mode function can be written as

$$\varphi_k(\tau) = \frac{H}{\sqrt{2c_s k^3}}\,(1 + ic_s k\tau)\, e^{-ic_s k\tau},$$

up to an irrelevant overall phase.

[*Hint: Show that the substitution* $u_k(\tau) = [\partial_\tau - 1/\tau] f_k(\tau)$ *in* $(*)$ *implies that* $f_k'' + c_s^2 k^2 f_k = 0$ *with general solution* $f_k(\tau) = A_k\, e^{ic_s k\tau} + B_k\, e^{-ic_s k\tau}$ *(or solve otherwise).*]

(ii) Discuss schematically the perturbative terms expected to appear when the action (†) is expanded to third order in φ. [You do not need to explicitly calculate their coefficients.] Briefly explain which of the resulting cubic terms will be expected to induce the largest contributions to non-Gaussianity.

Derive the coefficient λ multiplying the cubic term proportional to $\varphi\,(\varphi')^2$ in the third-order action from (†). Calculate the bispectrum $B(k_1,k_2,k_3)$ from $\lambda\varphi\,(\varphi')^2$ using the in-in formalism

$$\langle \hat{\varphi}_{\mathbf{k}_1}(0)\,\hat{\varphi}_{\mathbf{k}_2}(0)\,\hat{\varphi}_{\mathbf{k}_3}(0)\,\rangle = 2\,\mathrm{Im}\Big[\lim_{\tau\to 0}\Big\langle \hat{\varphi}_{\mathbf{k}_1}(\tau)\,\hat{\varphi}_{\mathbf{k}_2}(\tau)\,\hat{\varphi}_{\mathbf{k}_3}(\tau)\,\int_{-\infty(1-i\epsilon)}^{\tau} \mathcal{H}_I(\tau')d\tau'\Big\rangle\Big]\,,$$

where $\mathcal{H}_I$ is the interaction Hamiltonian.

2 For collisionless dark matter, the density perturbation $\delta(\mathbf{x},\tau)$ and the velocity $\mathbf{v}(\mathbf{x},\tau) = (v_1, v_2, v_3)$ in comoving coordinates, satisfy the continuity and Euler equations:

$$\delta' + \nabla\cdot[(1+\delta)\,\mathbf{v}] = 0\,, \quad v_i' + \mathcal{H}v_i + \mathbf{v}\cdot\nabla v_i = -\nabla_i\phi - \rho^{-1}\,\nabla_j\,(\rho\,\sigma_{ij}), \quad (i,j=1,2,3)\,, \quad (*)$$

where ρ is the matter density, σ_{ij} is the anisotropic stress tensor, and the gravitational potential ϕ obeys $\nabla^2\phi = \frac{3}{2}\mathcal{H}^2\delta$, where $\mathcal{H} = a'/a = 2/\tau$ in an EdS universe with $a(\tau)\propto\tau^2$. The linear solution is given by $\delta^{(1)}(\mathbf{k},\tau)\propto a$ with $\delta^{(1)\prime}(\mathbf{k},\tau) + \theta^{(1)}(\mathbf{k},\tau) = 0$ for $\theta\equiv\nabla\cdot\mathbf{v}$.

(i) Show how in Fourier space the nonlinear continuity equation takes the form

$$\delta'(\mathbf{k},\tau) + \theta(\mathbf{k},\tau) = -\int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3}\alpha(\mathbf{k}_1,\mathbf{k}_2)\,\theta(\mathbf{k}_1,\tau)\,\delta(\mathbf{k}_2,\tau)(2\pi)^3\delta^{(D)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}),$$

where $\alpha(\mathbf{k}_1,\mathbf{k}_2) = (\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_1/k_1^2$, which you should put in symmetric form. Find a similar expression for the Euler equation and show the coupling kernel is $\beta(\mathbf{k}_1,\mathbf{k}_2) = \frac{1}{2}(\mathbf{k}_1+\mathbf{k}_2)^2\,(\mathbf{k}_1\cdot\mathbf{k}_2)/(k_1^2k_2^2)$, stating the key assumptions made in these derivations.

Iterative solutions are sought order-by-order by using the power series ansatz,

$$\delta(\mathbf{k},\tau) = a(\tau)\,\tilde\delta^{(1)}(\mathbf{k}) + a^2(\tau)\,\tilde\delta^{(2)}(\mathbf{k}) + \ldots, \quad \theta(\mathbf{k},\tau) = -\mathcal{H}(\tau)[a\,\tilde\theta^{(1)}(\mathbf{k}) + a^2\,\tilde\theta^{(2)}(\mathbf{k}) + \ldots].$$

Show how the second-order density perturbation can be expressed in integral form as

$$\tilde\delta^{(2)}(\mathbf{k}) = \int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3}\,F_2(\mathbf{k}_1,\mathbf{k}_2)\,\tilde\delta^{(1)}(\mathbf{k}_1)\,\tilde\delta^{(1)}(\mathbf{k}_2)(2\pi)^3\delta^{(D)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}),$$

with the coupling kernel $F_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{5}{7}\,\alpha(\mathbf{k}_1,\mathbf{k}_2) + \frac{2}{7}\,\beta(\mathbf{k}_1,\mathbf{k}_2)$, where α,β are symmetrised.

(ii) Using diagrammatic methods or by a direct perturbative expansion, derive explicit integral representations for the one-loop matter power spectrum, giving your result in terms of the linear density field and the mode-coupling kernels $F_2(\mathbf{k}_1,\mathbf{k}_2)$ and $F_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$, with F_3 assumed to be also known. In particular, decompose the one loop power spectrum as $P_{\text{1-loop}}(k) = a^2P(k) + a^4\,[P_{22}(k) + 2P_{13}(k)]$, in terms of momentum-space integral expressions for the contributions from $P_{22}(k)$ and $P_{13}(k)$. [Do not attempt to integrate these.] Here, the linear power spectrum is given by $\langle\tilde\delta^{(1)}(\mathbf{k}_1)\,\tilde\delta^{(1)}(\mathbf{k}_2)\rangle = (2\pi)^3\delta^{(\mathrm{D})}(\mathbf{k}_1+\mathbf{k}_2)\,P(k)$.

(iii) Show that the $P_{22}(k)$ power spectrum can be written with $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$ in the form

$$P_{22}(k) = \int\frac{dq}{4\pi^2}\int_{-1}^{1}d\mu\,\frac{k^4\left(7k\mu + q(3-10\mu^2)\right)^2}{98\,(k^2-2kq\mu+q^2)^2}P(q)P\left(\sqrt{k^2-2\mu kq+q^2}\right). \qquad (\dagger)$$

Now assume a scale-free linear power spectrum $P(k) = Ak^n$. For the one-loop expression ($\dagger$), introduce the dimensionless ratio $r\equiv q/k$, and show that $P_{22}(k)$ can be factorized as $P_{22}(k) = A^2k^{2n+3}\,\mathcal{I}_{22}(n)$, where $\mathcal{I}_{22}(n)$ is a dimensionless integral over r and μ. In particular, derive the explicit dimensionless kernel $\mathcal{K}_{22}(r,\mu;n)$ such that

$$P_{22}(k) = A^2k^{2n+3}\int_0^\infty dr\int_{-1}^{1}d\mu\,\mathcal{K}_{22}(r,\mu;n).$$

Investigate the asymptotic limits $r\to0$, $r\to\infty$, and $r\to1$ and hence determine the conditions on the spectral index n for infrared and ultraviolet convergence.

3 This question focuses on the properties of biased tracers.

In the spherical collapse model, a region collapses to form a halo when δ_R exceeds the spherical-collapse threshold $\delta_c \simeq 1.686$, where δ_R is the linear density field smoothed on a scale R corresponding to mass $M = (4\pi/3)\bar{\rho}R^3$. The peak height ν is defined as $\nu \equiv \delta_c/\sigma(M)$. Here the smoothed linear density field has a variance $\sigma^2(M) = \langle \delta_R^2 \rangle$.

(a) The fraction of mass in collapsed objects with a mass greater than M is given naively by $F(> M) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\delta^2/(2\sigma^2)}\, \mathrm{d}\delta$. Following the Press-Schechter derivation by introducing an additional factor of 2, show that the halo mass function is given by:

$$\frac{\mathrm{d}n}{\mathrm{d}M} = \sqrt{\frac{2}{\pi}}\, \frac{\bar{\rho}}{M^2}\, \nu\, \exp\left(-\frac{\nu^2}{2}\right) \left|\frac{\mathrm{d}\ln\sigma}{\mathrm{d}\ln M}\right| . \tag{1}$$

(b) Now consider the clustering of halos in the peak background split formalism. By writing the density field as $\delta = \delta_s + \delta_l$ (where δ_s is the small-scale part and δ_l is a long-wavelength background perturbation) and noting that the effective collapse threshold becomes $\delta_c - \delta_l$ in the presence of δ_l, show that the linear Lagrangian bias is given by:

$$b_L(M) = \frac{\nu^2 - 1}{\delta_c} . \tag{2}$$

(c) The Eulerian bias $b(M) = 1 + b_L(M)$ must satisfy the consistency relation

$$\int_0^{\infty} \frac{\mathrm{d}n}{\mathrm{d}M}\, \frac{M}{\bar{\rho}}\, b(M)\, \mathrm{d}M = 1 . \tag{3}$$

Provide a physical justification for why this relation must hold. Verify that the Press–Schechter halo mass function and the bias from part **(b)** satisfy it.

(d) Assume that all matter resides in halos; each halo has a spherically symmetric density profile $\rho(r|M) = M\, u(r|M)$ with r the radius and u normalized such that $\int u\, \mathrm{d}^3x = 1$, where the integral is over all space. Let $\tilde{u}(k|M)$ be the Fourier transform of u. First, make the simplifying assumption that the locations of the halos are random. By writing the density field as a sum over halos, show that the matter power spectrum is given (with this simplifying assumption) by the one-halo term:

$$P(k) = P_{1h}(k) = \int \frac{\mathrm{d}n}{\mathrm{d}M} \left(\frac{M}{\bar{\rho}}\right)^2 |\tilde{u}(k|M)|^2\, \mathrm{d}M \tag{4}$$

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) Now, make the more realistic assumption that halos are not randomly distributed but are instead biased tracers of the long-wavelength density field as derived in part **(b)**. Under this assumption, derive that the power spectrum is given by $P(k) = P_{1h}(k)+P_{2h}(k)$, where $P_{1h}(k)$ is given by the expression in part (d) and $P_{2h}(k)$ is given by:

$$P_{2h}(k) = \left[\int \frac{\mathrm{d}n}{\mathrm{d}M}\ f(M,\bar{\rho})\ b(M)\ \tilde{u}(k|M)\ \mathrm{d}M\right]^2 P_{\mathrm{lin}}(k) \tag{5}$$

where you should specify the function $f(M,\bar{\rho})$.
[*Hint: you may assume that the expectation of a sum over distinct halo pairs with masses and positions* $(M_i,\vec{x}_i)$ *and* $(M_j,\vec{x}_j)$ *can be written as an integral over their joint distribution:*

$$\left\langle \sum_{i\neq j} g(M_i,\vec{x}_i,M_j,\vec{x}_j)\right\rangle = \int \mathrm{d}M_1\,\mathrm{d}M_2\,\mathrm{d}^3x_1\,\mathrm{d}^3x_2\,\frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2}$$
$$\times\left[1+b(M_1)\,b(M_2)\,\xi_{\mathrm{lin}}(|\vec{x}_1-\vec{x}_2|)\right] g(M_1,\vec{x}_1,M_2,\vec{x}_2)\,,$$

where ξ_{lin} *is the linear matter correlation function, given by* $\xi_{\mathrm{lin}}(|\vec{r}|) = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\,P_{\mathrm{lin}}(q)\,e^{i\vec{q}\cdot\vec{r}}$.]

4 In this question you will discuss acoustic oscillations in the photon–baryon plasma.

The Boltzmann equation for the photon temperature perturbation $\Theta(\eta, \mathbf{k}, \hat{\mathbf{p}})$ in Newtonian gauge can be written in Fourier space as:

$$\Theta' + ik\mu\,\Theta = \Phi' - ik\mu\Psi + \Gamma\left[\Theta_0 - \Theta + i\mu\, v_b\right], \tag{1}$$

where $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$, primes indicate partial derivatives with respect to conformal time η, $\Phi = \Psi$ are the Newtonian potentials, $\Gamma = a\sigma_T n_e$ is the Thomson scattering rate, and we have used $\mathbf{v_e}(\mathbf{k}) = \mathbf{v_b}(\mathbf{k}) = i\hat{\mathbf{k}}v_b$ (electron and baryon velocities are equal). The multipole expansion is $\Theta = \sum_l (-i)^l (2l+1)\,\Theta_l\, P_l(\mu)$. Finally, the baryon Euler equation is

$$v_b' = -\mathcal{H}v_b - k\Psi - \frac{\Gamma}{R}(3\Theta_1 + v_b) \tag{2}$$

where $R = 3\rho_b/(4\rho_\gamma) \propto a$ is the baryon–photon momentum–density ratio and $\mathcal{H}$ is the conformal Hubble parameter.

(a) Using the recurrence relation $(2l+1)\mu\, P_l = (l+1)P_{l+1} + l\, P_{l-1}$ and the orthogonality of Legendre polynomials $\int_{-1}^{1} P_l(\mu)P_{l'}(\mu)\, d\mu = \frac{2}{2l+1}\delta_{ll'}$, derive the photon continuity and Euler equations from the Boltzmann equation:

$$\Theta_0' = -k\Theta_1 + \Phi' \tag{3}$$

$$3\Theta_1' = k(\Theta_0 + \Psi) - \Gamma(3\Theta_1 + v_b) \tag{4}$$

where you may neglect the quadrupole Θ_2.

(b) In the tight-coupling regime $\Gamma \gg k \gtrsim \mathcal{H}$, the baryon and photon fluids move together. First use Eq. (4) to argue that $v_b = -3\Theta_1$ at leading order in Γ^{-1}. Then rearrange the baryon Euler equation (2) to obtain $3\Theta_1 + v_b$ to first order in Γ^{-1}, substituting the leading-order relation where needed. Finally, use this result and the continuity equation to show that the photon monopole satisfies

$$\Theta_0'' + \frac{\mathcal{H}R}{1+R}\,\Theta_0' + c_s^2 k^2\,\Theta_0 = -\frac{k^2}{3}\,\Psi + \Phi'' + \frac{\mathcal{H}R}{1+R}\,\Phi' \tag{5}$$

where $c_s^2 = 1/[3(1+R)]$ is the sound speed of the photon–baryon plasma.

(c) Neglecting baryons ($R = 0$) and time derivatives of the potentials, show that for adiabatic initial conditions the Sachs–Wolfe combination evolves as $(\Theta_0 + \Psi) \propto \cos(kr_s)$, where $r_s(\eta) = \int_0^\eta d\eta'\, c_s$.

(d) Consider the $l = 2$ Legendre transform of the Boltzmann equation. Show that in tight coupling the photon quadrupole is related to the dipole as follows:

$$\Theta_2 \approx C\frac{k}{\Gamma}\,\Theta_1 \tag{6}$$

for a constant C that you should determine.

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) The polarization of the CMB is produced by Thomson scattering of the local photon quadrupole Θ_2 at the last-scattering surface. Using your results, and assuming that the CMB polarization amplitude is $\propto \Theta_2$, show that the polarization amplitude at recombination is proportional to $\sin(kr_s^\star)$, where $r_s^\star$ is the sound horizon at recombination. Briefly discuss the consequences for the peak locations of CMB polarization power spectra relative to those in the CMB temperature power spectrum C_l^{TT}.

END OF PAPER