UNIVERSITY OF CAMBRIDGE

MAMA/311, NST3AS/311, MAAS/311

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 2:00pm to 5:00pm

PAPER 311

BLACK HOLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a null geodesic congruence, with U^a the unit tangent vector with respect to the affine parameter.

(i) Let $B_{ab} = \nabla_b U_a$. Explain why $U^a B_{ab} = B_{ab} U^b = 0$.

(ii) Let N^a be another null vector field satisfying $N^2 = 0$, $U \cdot N = -1$, and $U \cdot \nabla N^a = 0$. Use these to define a projector P^a_b which projects onto the 2-dimensional transverse subspace (i.e. the space perpendicular to both U and N).

(iii) Let $\hat{B}^a_b = P^a_c B^c_d P^d_b$. Explain how to define the *expansion* θ, *rotation* $\hat{\omega}_{ab}$ and *shear* $\hat{\sigma}_{ab}$ of the congruence in terms of $\hat{B}^a_b$.

(b) Consider a null geodesic congruence containing the generators of a null hypersurface $\mathcal{N}$. Show that the null hypersurface $\mathcal{N}$ satisfies the Raychaudhuri equation:

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - \hat{\sigma}_{ab}\hat{\sigma}^{ab} - R_{ab}U^a U^b.$$

[You may assume that the rotation $\hat{\omega}_{ab}$ vanishes because of the Frobenius theorem.]

(c) Assume the spacetime obeys the Einstein equation, with matter obeying the null energy condition. Show that if $\theta = \theta_0 < 0$ at a point p on a generator γ of a null hypersurface, then $\theta \to -\infty$ within finite affine parameter distance $2/|\theta_0|$, provided that γ extends this far.

(d) At late times, and over large enough distance scales, our own universe appears to be well-described by a flat expanding FLRW cosmology, with the metric

$$ds^2 = a(\eta)^2[-d\eta^2 + dx^2 + dy^2 + dz^2],$$

where $a(\eta)$ is a monotonically increasing function of the "conformal time coordinate" η. Suppose that at early times our universe is *not* accurately described by this metric, but nevertheless the complete spacetime $(\mathcal{M}, g)$ is a globally hyperbolic manifold with spatial topology $\mathbb{R}^3$, satisfying Einstein's equation and the null energy condition.

(i) Show that there exists an "anti-trapped" surface T in $(\mathcal{M}, g)$, i.e. the time-reverse of a trapped surface.

[You may use without proof $\theta = \dfrac{1}{\sqrt{h}}\dfrac{d\sqrt{h}}{d\lambda}$, where $h_{ab} = P^c_a P^d_b g_{cd}$ is the metric on the 2-dimensional transverse subspace.]

(ii) Argue that the universe is null-geodesically incomplete to the past. [You do not need to provide the full details of a rigorous proof. But for each restriction on $(\mathcal{M}, g)$ that is required for the conclusion to follow, your sketch of the argument should indicate how it is used.]

2

(a) Consider an eternal Schwarzschild black hole with mass $M > 0$. A spherically-symmetric shell of (radially) infalling radiation, with total energy $E > 0$, is dropped into the black hole from a sphere on past null infinity ($\mathcal{I}^-$), at some advanced time v. The shell of infalling radiation is nearly instantaneous (i.e. $\Delta v \approx 0$), but its energy E is not necessarily small compared to M. The shell eventually crosses the future horizon H^+ and hits the final singularity.

(i) Sketch a Penrose diagram for (the conformal compactification of) the spacetime described above. Indicate on your diagram the trajectory of the shell. Label the surfaces $\mathcal{I}^\pm$ and $H^\pm$ in your diagram (defined, in each case, with respect to the same asymptotically flat region that the shell falls in from). In the same asymptotic region, label past and future timelike infinity $i^\pm$, and spacelike infinity i^0.

(ii) Let λ be an affine parameter defined on the generators of H^+, scaled so that $\lambda = 0$ where H^+ intersects H^-, and $\lambda = 1$ at the moment the shell crosses H^+. Determine the intrinsic metric ds^2 on H^+, using the coordinates (λ, θ, ϕ), for both $\lambda > 1$ and $\lambda < 1$.

(iii) What is the allowed range of the λ coordinate on H^+? Explain your answer physically.

[Hint: after solving (iii) you may wish to check the Penrose diagram you drew in part (i) to make sure it is compatible with what you have shown here.]

(b) Suppose now that multiple shells of spherically-symmetric energy are dropped from $\mathcal{I}^-$ into the horizon at different advanced times v. By considering the stress-tensor on the horizon, show that when the total energy E of all the shells is small compared to M, the physical-process version of the First Law of black hole mechanics $\Delta M = T\Delta S$ is satisfied.

3

(a) The Schwarzschild metric has the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \qquad f(r) = 1 - \frac{2M}{r}.$$

Although the metric has a coordinate singularity at $r = 2M$, nevertheless for $0 < r < 2M$, this metric gives a valid description of either region II (inside the black hole horizon) or region III (inside the white hole horizon), depending on the choice of time-orientation. Assume we have selected the time-orientation appropriate to region II.

In this region only, answer the following:

(i) Identify four linearly independent Killing vectors for the metric. Show that all Killing vectors in the region are spacelike.

(ii) Show that every future-directed timelike curve (starting from a point p in the region) has monotonically decreasing r. Show that it intersects the singularity at $r = 0$ within a proper time τ less than πM.

(b) In a 1+1 dimensional Minkowski spacetime with no gravity, with the usual coordinates $ds^2 = -dt^2 + dx^2$, the matter stress-tensor at a point p takes the form

$$T_{\mu\nu} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Write down conditions on a, b, c for this stress-energy tensor to satisfy:

(i) the null energy condition (NEC);

(ii) the dominant energy condition (DEC);

(iii) the weak energy condition (WEC), in the case where $c > a$.

(c) Consider a massless real scalar field Φ on a globally hyperbolic spacetime $(\mathcal{M}, g)$ with the action:

$$S = -\int_{\mathcal{M}} d^4\mathbf{x}\sqrt{-g}\,\partial_a\Phi\partial^a\Phi.$$

(i) Using a 3+1 decomposition of the spacetime, write down the canonical commutation relations for the scalar field. Define the Klein-Gordon norm on the space of complex solutions, and explain why it does not depend on the choice of foliation of $(\mathcal{M}, g)$.

(ii) Supposing that $\mathcal{M}$ is strictly stationary, explain how to uniquely define a vacuum state. [You do not need to derive the commutation relations for a and $a^\dagger$, but you do need to say how these operators are defined.] Hence, construct the Fock space on $\mathcal{M}$. Why does this construction fail if $\mathcal{M}$ is not stationary?

4

A rotating Kerr black hole may be described by the Boyer-Lindquist metric:

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}[a\, dt - (r^2 + a^2)d\phi]^2,$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. You may assume that the black hole is subextremal with $0 < a < M$.

(a) (i) What is a *stationary* black hole metric? What additional condition must it satisfy to be *static*?

(ii) What is a *stationary and axisymmetric* black hole metric?

(iii) Show that the Kerr black hole is stationary and axisymmetric, but not static.

(b) Calculate the coordinate location of the ergoregion. Compare this to the location of the event horizon. Briefly explain why this permits energy extraction from the Kerr black hole.

(c) In this part you will consider only geodesics which are confined to the north pole $\theta = 0$ at arbitrary values of (r, t).

(i) Find the pullback of the Boyer-Lindquist metric to the 1+1 dimensional surface with $\theta = 0$, i.e. the metric intrinsic to the surface.

(ii) Write a differential equation for the tortoise (or Regge-Wheeler) coordinate r_*. Use this to rewrite the resulting metric in terms of a retarded time coordinate u, and thereby give an analytic extension to the white hole region of Kerr.

(iii) What happens at $r = 0$, and what is the maximum range of the r coordinate?

(d) On an asymptotically flat hypersurface Σ with asymptotically Cartesian coordinates, the ADM energy $E_{\rm ADM}$ is given by

$$E_{\rm ADM} = \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r^2} dA\, n_i(\partial_j h_{ij} - \partial_i h_{jj}),$$

where h_{ij} is the metric on Σ, and repeated indices are summed. Calculate $E_{\rm ADM}$ for the Kerr black hole (in its rest frame), thus verifying that M may be interpreted as the mass of the black hole.

END OF PAPER