UNIVERSITY OF CAMBRIDGE

MAMA/310, NST3AS/310, MAAS/310

MAT3

MATHEMATICAL TRIPOS Part III

Monday 15 June 2026 9:00am to 12:00pm

PAPER 310

COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a particle species X of mass m, temperature T, with g internal degrees of freedom, and chemical potential μ.

(i) Write down the equilibrium phase-space distribution function $f(p)$ for X, clearly distinguishing between bosons and fermions.

(ii) Starting from $f(p)$, write integral expressions for the number density n, energy density ρ, and pressure p of the species X.

(iii) Approximate these integral expressions in the relativistic limit $T \gg m$ neglecting the chemical potential and hence derive the temperature dependence of ρ, p and n.

(b) The species X remains in thermal equilibrium with the standard model plasma until it decouples instantaneously at temperature T_{dec}. Throughout this question, you may assume X is relativistic, $m \simeq 0$ and you may neglect the chemical potential. Let $g_\star$ be the effective number of relativistic degrees of freedom of the plasma at T_{dec}.

(i) Define the entropy density s and state the conservation equation it obeys.

(ii) Assuming that the only relativistic degrees of freedom at late times are the photon and X, compute the temperature T_X of X today in terms of that of the photon T_γ.

2

Consider slow-roll inflation driven by the scalar field ϕ subject to the so-called hill-top potential

$$V = \Lambda^4 - \frac{1}{2} m^2 \phi^2 \,,$$

where Λ and m are constants. Throughout this question you may assume $\phi > 0$ and approximate your results to the leading non-trivial order in $\Lambda^4 \gg m^2\phi^2$.

(a) Compute the potential slow-roll parameters

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \text{and} \qquad \eta_V \equiv M_{\rm Pl}^2 \frac{V''}{V} \,.$$

(b) Using the appropriate value of ϵ_V, estimate the value ϕ_e of the scalar field at the end of inflation.

(c) Write down the background equation of motion for ϕ and the Friedmann equation to leading order in slow roll.

(d) In the slow-roll approximation, compute the ϕ-dependence of the number of efoldings $N(\phi)$ before the end of inflation, so that $N(\phi_e) = 0$. Hence derive an expression for ϕ_{60}, namely the value of ϕ 60-efoldings before the end of inflation.

(e) Recalling that $n_s - 1 \simeq -6\epsilon_V + 2\eta_V$ and

$$P_{\mathcal{R}} = \frac{H^2}{4\epsilon M_{\rm Pl}^2}\frac{1}{k^3} = \frac{2\pi^2 A_s}{k^3}\left(\frac{k}{k_\star}\right)^{n_s - 1} \,,$$

derive an expression of the primordial amplitude A_s and the spectral tilt $n_s - 1$ in terms of ϕ and the parameters Λ and m of the model.

(f) Imposing $A_s = 2\times 10^{-9}$ and $n_s - 1 = -1/30$ at $\phi = \phi_{60}$, derive an order-of-magnitude estimate for the scale of inflation in Planck units, $H/M_{\rm Pl}$.

3

Consider a perturbed flat FLRW universe with scale factor $a(t)$ and total energy momentum tensor $T_{\mu\nu}$. In this question, you will derive the scalar part of the equation for the conservation of momentum in synchronous gauge.

(a) State the covariant conservation of the energy-momentum tensor.

(b) Working to linear order in perturbations around the homogeneous background, where $\bar{T}^{\mu}_{\ \nu} = \mathrm{diag}(-\bar{\rho}, \bar{p}, \bar{p}, \bar{p})$, show that

$$\delta(\nabla_\mu T^{\mu}_{\ \nu}) = \partial_\mu \delta T^{\mu}_{\ \nu} + \bar{\Gamma}^{\mu}_{\mu\lambda}\, \delta T^{\lambda}_{\ \nu} - \bar{\Gamma}^{\lambda}_{\mu\nu}\, \delta T^{\mu}_{\ \lambda} + \delta\Gamma^{\mu}_{\mu\lambda}\, \bar{T}^{\lambda}_{\ \nu} - \delta\Gamma^{\lambda}_{\mu\nu}\, \bar{T}^{\mu}_{\ \lambda}.$$

(c) You may now use that the background Christoffel symbols for a spatially flat FLRW universe in cosmic time are

$$\bar{\Gamma}^{0}_{ij} = a^2 H\, \delta_{ij}, \qquad \bar{\Gamma}^{i}_{0j} = H\, \delta_{ij},$$

with all others vanishing or related by symmetry. Here $H = \dot{a}/a$. Using the result above, show that the spatial component ($\nu = j$) of the conservation equation can be written as

$$\partial_0 \delta T^{0}_{\ j} + \partial_i \delta T^{i}_{\ j} + 2H\, \delta T^{0}_{\ j} - a^2 H\, \delta T^{j}_{\ 0} - (\bar{\rho} + \bar{p}) \left(\frac{1}{2} \partial_j h_{00} - H h_{j0} \right) = 0,$$

where $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ are metric perturbations around the FLRW metric $\bar{g}_{\mu\nu}$.

(d) In synchronous gauge the scalar perturbations of the metric are defined by $h_{0\mu} = 0$, so that the perturbed metric takes the form

$$ds^2 = -dt^2 + a^2(t)\, dx^i dx^j \left[\delta_{ij}(1 + A) + \partial_i \partial_j B \right].$$

For a generic fluid, the scalar velocity perturbation and possible scalar anisotropic stress π^S are defined through

$$\delta T^{0}_{\ i} = (\bar{\rho} + \bar{p})\, \partial_i \delta u, \qquad\qquad \delta T^{i}_{\ j} = \delta p\, \delta^{i}_{\ j} + \partial_i \partial_j \pi^S.$$

In this gauge, simplify the momentum conservation equation derived in Part (c).

(e) Extract its scalar (longitudinal) part and show that the Euler equation becomes

$$\delta p + \nabla^2 \pi^S + \partial_0 [(\bar{\rho} + \bar{p}) \delta u] + 3H (\bar{\rho} + \bar{p}) \delta u = 0.$$

(f) Is it consistent that metric perturbations do not appear explicitly in this equation in synchronous gauge? Briefly comment on this and contrast it with the Euler equation derived in Newtonian gauge.

4

Newtonian perturbation theory is described by the following equations:

$$\dot{\delta} = -\frac{1}{a}\nabla \cdot \mathbf{v}\,, \qquad \dot{\mathbf{v}} + H\mathbf{v} = -\frac{1}{a\bar{\rho}}\nabla\delta p - \frac{1}{a}\nabla\phi\,, \qquad \nabla^2\phi = 4\pi G a^2 \bar{\rho}\,\delta\,,$$

where $\mathbf{v}(\mathbf{x},t)$ is the peculiar velocity, $\phi(\mathbf{x},t)$ the gravitational potential and $\delta(\mathbf{x},t)$ is the matter density contrast for dark matter with background energy density $\bar{\rho}$.

(a) Assuming that the vorticity of the velocity field can be neglected, extract from the Euler equation an evolution equation for the velocity divergence $\theta \equiv \nabla \cdot \mathbf{v}$.

(b) Combine the three equations above to obtain a single second-order differential equation for the density contrast δ.

(c) Now assume a barotropic equation of state $p = p(\rho)$ and define the sound speed

$$c_s^2 \equiv \frac{\partial p}{\partial \rho}.$$

Discuss the notion of Jeans instability and the definition of the Jeans wavenumber k_J. Discuss qualitatively the behavior of modes with $k \gg k_J$, and $k \ll k_J$.

(d) Now consider the case of cold dark matter, for which $c_s = 0$. Specialize the equation obtained above to this case and derive the evolution equation for matter perturbations during a matter-dominated universe. Solve the equation and determine the leading growing mode for $\delta(t)$.

(e) Now imagine that the universe is dominated by a fluid with energy density $\rho_D \gg \rho_m$ and equation of state parameter $w = -2/3$, which has negligible inhomogeneities, $\rho_D(t,\mathbf{x}) = \bar{\rho}_D(t)$.

(i) State or derive the time dependence of the Hubble parameter and scale factor in this case.

(ii) Specialize the result of (b) to this case and solve the resulting differential equation for $\delta(t)$ at late times, $a \gg 1$.

END OF PAPER