UNIVERSITY OF CAMBRIDGE

MAMA/309, NST3AS/309, MAAS/309

MAT3

**MATHEMATICAL TRIPOS Part III**

Friday 5 June 2026 9:00am to 12:00pm

# PAPER 309

# GENERAL RELATIVITY

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1 General Relativity**

Let $(M, g)$ be a smooth, 4-dimensional Lorentzian manifold, and throughout this question we take $\nabla$ to be the Levi–Civita connection.

(a) Define what it means for a vector field $K$ to be a *Killing* vector field. Let $\gamma$ be an affinely parametrized geodesic with tangent vector $U$. Show that $g(K, U)$ is constant along $\gamma$. Explain briefly how this result generalises the conservation laws associated with symmetries in Minkowski space.

(b) The *Lie derivative* $\mathcal{L}_X\nabla$ *of the Levi–Civita connection along a vector field* $X$ can be defined by

$$(\mathcal{L}_X\nabla)_Y Z = \mathcal{L}_X(\nabla_Y Z) - \nabla_{\mathcal{L}_X Y}(Z) - \nabla_Y(\mathcal{L}_X Z)$$

for all smooth vector fields $X$, $Y$ and $Z$. Verify that

$$(\mathcal{L}_X\nabla)_Y(fZ) = f\,(\mathcal{L}_X\nabla)_Y Z$$

for any smooth function $f$. Show that

$$(\mathcal{L}_X\nabla)_Y Z = R(X,Y)Z + \nabla_Y(\nabla_Z X) - \nabla_{\nabla_Y Z}X\,,$$

where $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the Riemann tensor of $\nabla$. [*Hint: Recall that the Levi–Civita connection $\nabla$ is torsion-free.*]

(c) Briefly explain why $\mathcal{L}_K\nabla = 0$ for any Killing vector $K$. Hence or otherwise show that

$$\nabla_U\nabla_U K = R(U,K)U$$

where $K$ and $U$ are as in part (a). Briefly discuss the physical interpretation of this equation in terms of geodesic deviation.

**2 General Relativity**

Let $(\mathbb{R}^4, \eta)$ be four-dimensional Minkowski space-time. Consider a vacuum linearised metric perturbation

$$g_{ab} = \eta_{ab} + h_{ab}\,, \qquad \text{where} \qquad |h_{ab}| \ll 1\,,$$

and let $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h$ with $h = h^c{}_c$. (Throughout this question, indices are raised and lowered using the background Minkowski metric.)

(a) How do linearised gauge transformations (linearised diffeomorphisms) act on $\bar{h}_{ab}$? Show that we can use these to achieve Lorenz gauge $\partial^a \bar{h}_{ab} = 0$. Show that there are residual gauge transformations, preserving Lorenz gauge.

(b) Show that the linearised field equations $\partial^c \partial_c \bar{h}_{ab} = 0$ and Lorenz gauge condition are solved by a (complex) plane wave $\bar{h}_{ab}(x) = A_{ab}\, e^{ik\cdot x}$, provided the constant covector $k_a$ and constant polarization tensor $A_{ab}$ obey appropriate constraints which you should specify. Find how the residual transformations act on $A_{ab}$ and show they may be used to impose $A^a{}_a = 0$.

(c) Let

$$\mathcal{V} = k^\perp \text{ considered mod } \langle k \rangle\,,$$

where $k^\perp = \{v^a : k_a v^a = 0\}$ and $\langle k \rangle$ denotes the line spanned by $k^a$. Prove that $A_{ab}$ determines a gauge invariant, trace-free, symmetric bilinear form on $\mathcal{V}$.

(d) Define a linear map $\mathcal{R} : \Lambda^2 \to \Lambda^2$ on the space $\Lambda^2$ of 2-forms by

$$(\mathcal{R}(\omega))_{ab} = \tfrac{1}{2} R^{(1)cd}_{ab}\, \omega_{cd}$$

for any 2-form $\omega$, where $R^{(1)cd}_{ab}$ is the linearized Riemann curvature. For a plane wave solution as in part (b), show that

$$\mathcal{R}(k \wedge v) = 0 \qquad \text{for all vectors } v,$$

where $(k \wedge v)_{cd} = k_c v_d - v_c k_d$ with $v_d = \eta_{de} v^e$. Also show that

$$\mathcal{R}(\omega) = 0$$

for all 2-forms $\omega$ which obey $\omega_{ab} k^b = 0$. Deduce a bound on the rank (*i.e.* dimension of the image) of the linear map $\mathcal{R}$ at each point $p \in \mathbb{R}^4$.

**3 General Relativity**

Let $(M, g)$ be a 4 dimensional space-time with Lorentzian metric $g$.

(a) Define what is meant by an *orthonormal frame* $\{e_a\}$. To what extent is such a frame determined uniquely by $g$? How is the dual coframe $\{\theta^b\}$ defined? How is $g$ related to this coframe?

(b) Define the *connection 1-forms* $\omega^a{}_b$ and the *curvature 2-forms* $\mathcal{R}^a{}_b$ associated to the coframe $\theta^a$. How is $\mathcal{R}^a{}_b$ related to the Riemann curvature tensor?

(c) Consider a metric of the form

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\phi^2) \qquad (\dagger)$$

where $f(r) > 0$ for all $r > 0$. Construct an orthonormal coframe and compute the curvature 2-forms for this metric.

(d) The *null energy condition* states that the energy momentum tensor $T_{ab}$ should obey $T_{ab}k^a k^b \geqslant 0$ for any null vector $k^a$. Suppose the metric ($\dagger$) obeys Einstein's equations for a matter distribution obeying the null energy condition. By considering a null vector with non-zero angular component, show that

$$f'' \geqslant \frac{2(f-1)}{r^2}.$$

**4 General Relativity**

Let $(M, g)$ be a 4-dimensional oriented manifold with a Lorentzian metric $g$, and let $\Omega^p(M)$ denote the space of smooth $p$-forms on $M$.

(a) What is meant by a *diffeomorphism* $\varphi$ of $M$? Suppose $\omega \in \Omega^p(M)$. How do diffeomorphisms act on $\omega$? Show that this action commutes with taking the exterior derivative $d$ of $\omega$.

(b) Define the *Hodge star operator* $* : \Omega^p(M) \to \Omega^{4-p}(M)$ in terms of the metric $g$. Show that when acting on 2-forms, $*$ is invariant under $g \mapsto e^{2\psi} g$ for any smooth function $\psi$.

(c) The action for gravity coupled to Maxwell theory can be written as

$$S[g, A] = \frac{1}{16\pi G_{\mathrm{N}}} \int_M R_g \,\mathrm{vol}_g + \frac{1}{2} \int_M F \wedge *F$$

where $R_g$ is the Ricci scalar of the metric $g$, $\mathrm{vol}_g$ is the volume form and $F = dA$ is the electromagnetic field strength. Show that this action is invariant under any (orientation preserving) diffeomorphism $\varphi$.

(d) Under a generic infinitesimal variation of the metric $g$ and gauge potential $A$, the action changes as

$$\delta S[g, A] = \int_M \left( E^{\mu\nu}\, \delta g_{\mu\nu} + \mathcal{E}^{\mu}\, \delta A_{\mu} \right) \mathrm{vol}_g \ + \ \text{(boundary term)}.$$

Assuming suitable boundary conditions, show that diffeomorphism invariance implies the Noether identity

$$\nabla_{\mu} E^{\mu}{}_{\nu} + \frac{1}{2}\, \mathcal{E}^{\mu} F_{\mu\nu} + \frac{1}{2}\, A_{\nu}\, \nabla_{\mu} \mathcal{E}^{\mu} \ = \ 0.$$

Deduce that if the Maxwell equations hold then

$$\nabla_{\mu} E^{\mu}{}_{\nu} = 0.$$

*[Hint: You may assume that, under an infinitesimal diffeomorphism generated by a vector field $X$, the change in $A_\mu$ can be written as $(\mathcal{L}_X A)_\mu = X^\nu F_{\nu\mu} + \nabla_\mu (X^\nu A_\nu)$.]*

## END OF PAPER