UNIVERSITY OF CAMBRIDGE

MAMA/306, NST3AS/306, MAAS/306

MAT3

MATHEMATICAL TRIPOS Part III

Friday 12 June 2026 9:00am to 12:00pm

PAPER 306

STRING THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

The following conventions apply to ALL questions

The motion of the bosonic string in D-dimensional Minkowski space is described by an embedding map,

$$X : \Sigma \quad \rightarrow \quad \mathbb{R}^{1,D-1}$$

which takes a point (σ, τ) on the string worldsheet Σ to a point $X^\mu(\sigma, \tau)$ in space time. We also introduce a dynamical metric $g_{\alpha\beta}(\sigma, \tau)$ with Lorentzian signature on Σ. String motion is governed by the Polyakov action,

$$S_P = -\frac{T}{2} \int_\Sigma d^2\sigma \, \sqrt{-g} g^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X$$

where g denotes the determinant of the 2×2 matrix $g_{\alpha\beta}$ and, for any two Lorentz D-vectors X and Y, $X \cdot Y := \eta_{\mu\nu} X^\mu Y^\nu$. Here,

$$\eta \quad = \quad \mathrm{diag}(-1, +1, \ldots, +1)$$

is the flat metric on $\mathbb{R}^{1,D-1}$.

1

(a) Define the *induced metric* $\gamma_{\alpha\beta}$ on the string worldsheet Σ. Write down the *Nambu-Goto action* for the string. What is its geometric interpretation? Show that it is equivalent to the Polyakov action.

(b) By applying the principle of least action directly to the Nambu-Goto action show that the equation of motion for $X^\mu(\sigma, \tau)$ takes the form,

$$\partial_\alpha \left(\sqrt{-\det\gamma}\, \gamma^{\alpha\beta} \partial_\beta X^\mu \right) \quad = \quad 0.$$

(c) In the following we set $X^0 = t$, $X^1 = x$ and $X^2 = y$.

(i) Working in a static gauge where $t = X^0 = \tau$, find a solution of this equation of motion corresponding to a circular loop of closed string in the x-y plane with time dependent radius $R(\tau)$ and initial conditions, $R(0) = R_0$, $\dot{R}(0) = 0$.

(ii) Evaluate the induced metric $\gamma_{\alpha\beta}$ on this solution.

[*Hint: In this question you may use without proof the following result for the variation of the determinant of a symmetric matrix A:* $\delta(\det A) = \det A\, \mathrm{Tr}(A^{-1}\delta A)$]

2

(a) Describe the residual gauge invariance of the conformal gauge Polyakov action. Define the condition for *lightcone gauge* and show that it fixes this residual gauge invariance.

(b) Describe the quantization of the closed string in lightcone gauge, giving a description of string states in terms of creation and annihilation operators. You should explain how any normal ordering ambiguity is resolved by imposing necessary conditions for Lorentz invariance.

(c) Hence determine the mass spectrum of the closed string and find the total degeneracy of the each of the first four levels in D spacetime dimensions.

3

(a) By giving its transformation property under a finite conformal transformation, explain what is meant by a *primary operator* $O(z,\bar{z})$ of conformal weight $(h,\tilde{h})$ in a two-dimensional conformal field theory. Give an alternative definition by stating the singular terms in the operator product $T(z)O(w,\bar{w})$ where $T(z)$ is the holomorphic stress-energy tensor.

(b) Now consider the theory of free scalar field $X(z,\bar{z})$ with action,

$$S \;=\; \frac{1}{2\pi}\int d^2z\,\partial X\bar{\partial}X.$$

Derive a formula for the two-point function $\langle X(z,\bar{z})X(w,\bar{w})\rangle$. [You may use without proof the relation $\partial\bar{\partial}\log|z-w|^2 = 2\pi\delta(z-w,\bar{z}-\bar{w})$]

(c) In the following, we define the operator,

$$O_k(z,\bar{z}) \;=\; :\exp\left(ikX(z,\bar{z})\right):,$$

where k is a real parameter, $:\ldots:$ denotes normal ordering and the exponential is defined via its Taylor expansion.

(i) By considering its OPE with holomorphic stress tensor $T(z)$, show that $O_k(z,\bar{z})$ is a primary operator and determine its conformal weight.

(ii) Using Wick's theorem and the Taylor expansion of exponential, compute the correlation function,

$$G_{k,-k} \;:=\; \langle O_k(z,\bar{z})O_{-k}(w,\bar{w})\rangle.$$

By considering the behaviour of $G_{k,-k}$ under a scaling transformation $z\to\lambda z$, $\bar{z}\to\lambda\bar{z}$, for some real number $\lambda>0$, check whether your results for parts (i) and (ii) are consistent with each other.

4 The Polyakov path integral for the partition function of bosonic string theory has the formal expression,

$$Z \quad = \quad \int \frac{\mathcal{D}X\mathcal{D}g}{\mathrm{Vol}[\mathcal{G}]} \exp\left(-S_P[X,g]\right),$$

where $\mathrm{Vol}[\mathcal{G}]$ denotes the volume of the infinite-dimensional gauge group generated by diffeomorphisms and Weyl transformations of the string worldsheet and $S_P[X,g]$ denotes the Polyakov action.

(a) Explain in detail how you can eliminate the volume of the gauge orbit by introducing anti-commuting ghost fields c_α and $b_{\alpha\beta}$ on the string worldsheet to express the partition function in the form,

$$Z \quad = \quad \int \mathcal{D}X\,\mathcal{D}c\mathcal{D}b \exp\left(-S_P[X,g] - S_{Gh}[b,c,g]\right),$$

where you should give the action S_{Gh} for the ghost fields explicitly.

(b) Working in conformal gauge and in Euclidean signature on the worldsheet, evaluate the two-point correlation function of ghost fields, $\langle b_{\alpha\beta}(\sigma,\tau)c_\delta(\sigma',\tau')\rangle$. You should express your answer in complex worldsheet coordinates.

END OF PAPER